Chapter 3: Wave Properties of Particles

- 1905 discovery of the particle properties of waves: A revolutionary concept to explain data.
- 1924 (Louis de Broglie's PhD thesis) speculation that particles might show wave behavior: An equally revolutionary concept without a strong experimental mandate.
    - moving objects have wave as well as particle characteristics
- 1927 The existence of de Broglie waves was experimentally demonstrated.
- The duality principle provided the starting point for Schrödinger's successful development of quantum mechanics.

3.1 DE BROGLIE WAVES

- A moving body behaves in certain ways as though it has a wave nature.
- A photon of light of frequency ν has the momentum

$$p = \frac{h\nu}{c} = \frac{h}{\lambda}$$

Photon wavelength $$\lambda = \frac{h}{p} \tag{3.1}$$

- De Broglie suggested that Eq. (3.1) is a completely general one that applies to material particles as well as to photons.
- Part of de Broglie's inspiration came from Bohr's theory of the hydrogen atom, in which the electron is supposed to follow only certain orbits around the nucleus.
- Two years later Erwin Schrödinger used the concept of de Broglie waves to develop a general theory that he and others applied to explain a wide variety of atomic phenomena.
- The existence of de Broglie waves was confirmed in diffraction experiments with electron beams in 1927.
- The momentum of a particle of mass m and velocity v is p=γmv, and its **de Broglie wavelength** is accordingly

Louis de Broglie 1892-1987 Nobel Prize in Physics in 1929

De Broglie wavelength $$\lambda = \frac{h}{\gamma m v} \tag{3.2}$$

γ is the relativistic factor $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

- In certain situations, a moving body resembles a wave and in others it resembles a particle.
    - Which set of properties is most apparent depends on how its de Broglie wavelength compares with its dimensions and the dimensions of whatever it interacts with.

Example 3.1&3.2

Find the de Broglie wavelengths of (a) a 46-g golf ball with a velocity of 30 m/s, and (b) an electron with a velocity of 10^7 m/s.

Find the kinetic energy of a proton whose de Broglie wavelength is 1.000 fm=1.000x10^{-15} m, which is roughly the proton diameter.

3.2 WAVES OF WHAT?

- Waves of **probability**.
    - In water waves, the quantity that varies periodically is the height of the water surface.
    - In sound waves, it is pressure.
    - In light waves, electric and magnetic fields vary.
    - What is it that varies in the case of matter waves?

- The quantity whose variations make up matter waves is called the wave function, symbol Ψ.
- Born came the basic concept that the wave function Ψ of a particle is probability of finding it.
- The wave function Ψ itself, however, has no direct physical significance. By itself cannot be an observable quantity.
- $|\Psi|^2$ must represent probability density for electrons (or other particles).
- For this purpose, atomic scattering (collisions of atoms with various particles) processes suggested.

Max Born
1882–1970
Nobel Prize in Physics in 1954

- $|\Psi|^2$; the square of the absolute value of the wave function, which is known as probability density.

- The probability of experimentally finding the body described by the wave function at the point x, y, z, at the time t is proportional to the value of $|\Psi|^2$ there at t.
  - A large value of $|\Psi|^2$ means the strong possibility of the body's presence,
  - while a small value of $|\Psi|^2$ means the slight possibility of its presence.
- There is a big difference between the probability of an event and the event itself.
  - Although we can speak of the wave function Ψ that describes a particle as being spread out in space, this does not mean that the particle itself is thus spread out.
  - When an experiment is performed to detect electrons, for instance, a **whole** electron is either found at a certain time and place or it is not; there is no such thing as a 20 percent of an electron.
  - However, it is entirely possible for there to be a 20 percent **chance** that the electron be found at that time and place, and it is this likelihood that is specified by $|\Psi|^2$.
- While the wavelength of the de Broglie waves associated with a moving body is given by the simple formula $\lambda = h/\gamma mv$, to find their amplitude Ψ as a function of position and time is often difficult.

3.3 DESCRIBING WAVE

- A general formula for waves.
- How fast do de Broglie waves travel? Since we associate a de Broglie wave with a moving body, we expect that this wave has the same velocity as that of the body??
- Call the de Broglie wave velocity as $v_p = v$

De Broglie phase velocity

$$v_p = \nu\lambda = \left(\frac{\gamma mc^2}{h}\right)\left(\frac{h}{\gamma mv}\right) = \frac{c^2}{v} \qquad (3.3)$$

  - Because the particle velocity v must be less than the velocity of light c, the de Broglie waves always travel faster than light!
  - In order to understand this unexpected result, we must look into the distinction between phase velocity (wave velocity) and group velocity.
- How waves are described mathematically?

- Fig. 3.1. If we choose t=0 when the displacement y of the string at x=0 is a maximum, its displacement at any future time t at the same place is given by the formula

$$y = A \cos 2\pi\nu t \qquad (3.4)$$

  - where A is the amplitude of the vibrations (that is, their maximum displacement on either side of the x axis) and ν their frequency.
  - Equation (3.4) tells us what the displacement of a single point on the string is as a function of time t.

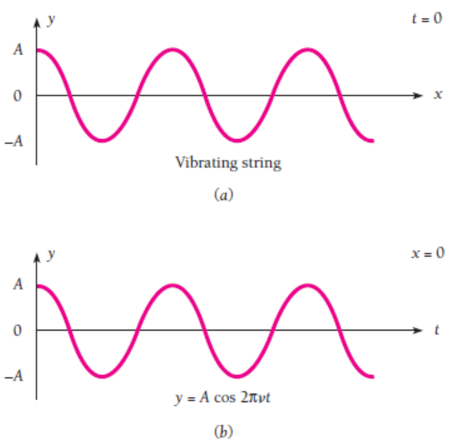


Figure 3.1 (a) The appearance of a wave in a stretched string at a certain time. (b) How the displacement of a point on the string varies with time.

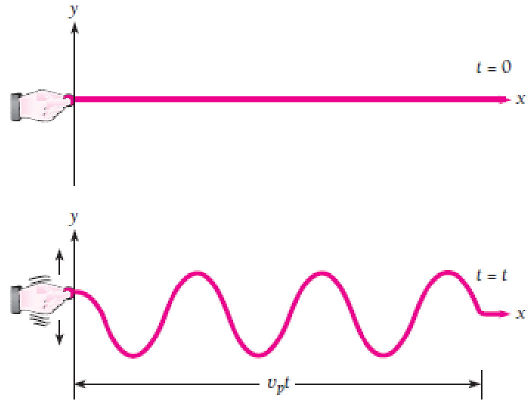


Figure 3.2 Wave propagation

- A complete description of wave motion in a stretched string, however, should tell us what y is at <u>any point</u> on the string at <u>any time.</u>
- To obtain such a formula, let us imagine that we shake the string at x=0 when t=0, so that a wave starts to travel down the string in the +x direction (Fig. 3.2).

Wave formula
$$y = A \cos 2\pi\nu\left(t - \frac{x}{v_p}\right) \qquad (3.5)$$

Since the wave speed v_p is given by $v_p=\nu\lambda$, we have

Wave formula
$$y = A \cos 2\pi\left(\nu t - \frac{x}{\lambda}\right) \qquad (3.6)$$

- Most widely used description of a wave, however, is still another form of Eq. (3.5). The quantities angular frequency ω and wave number k are defined by the formulas.

Angular frequency
$$\omega = 2\pi\nu \qquad (3.7)$$

Wave number
$$k = \frac{2\pi}{\lambda} = \frac{\omega}{v_p} \qquad (3.8)$$

  - The unit of ω is the radian per second and that of k is the radian per meter.
  - Angular frequency gets its name from uniform circular motion, where a particle that moves around a circle ν times per second sweeps out 2πν rad/s.
  - The wave number is equal to the number of radians corresponding to a wave train 1 m long, since there are 2π rad in one complete wave.

Wave formula
$$y = A \cos(\omega t - kx) \qquad (3.9)$$

3.4 PHASE AND GROUP VELOCITIES

- A group of waves need not have the same velocity as the waves themselves.
- The amplitude of the de Broglie waves that correspond to a moving body reflects the probability that it will be found at a particular place at a particular time.
- de Broglie waves **cannot** be represented simply by a formula resembling Eq. (3.9), which describes an indefinite series of waves all with the same amplitude A.
- Instead, we expect the wave representation of a moving body to correspond to a **wave packet**, or **wave group**, like that shown in Fig. 3.3, whose waves have amplitudes upon which the likelihood of detecting the body depends.

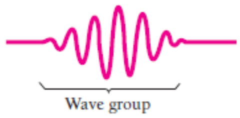
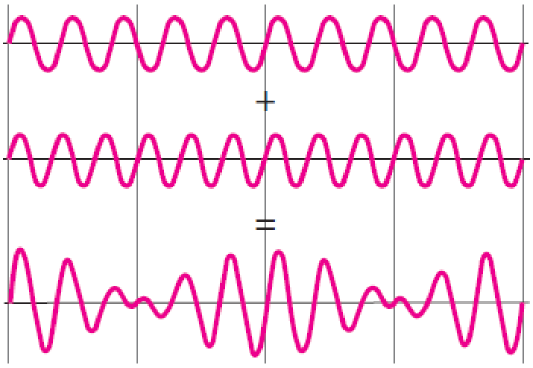


Figure 3.3 A wave group.



- A familiar example of how wave groups come into being is the case of beats.
- If the original sounds have frequencies of, say, 440 and 442 Hz, we will hear a fluctuating sound of frequency 441 Hz with two loudness peaks, called beats, per second.
- The production of beats is illustrated in Fig. 3.4.

Figure 3.4 Beats are produced by the superposition of two waves with different frequencies

- A wave group: a superposition of individual waves of different wavelengths whose interference with one another results in the variation in amplitude that defines the group shape.
  - If the velocities of the waves are the same, the velocity with which the wave group travels is the common phase velocity.
  - However, if the phase velocity varies with wavelength, the different individual waves do not proceed together. This situation is called **dispersion**.
  - **As a result**, the wave group has a velocity different from the phase velocities of the waves that make it up. This is the case with de Broglie waves.

De Broglie group velocity

$$v_g = v \qquad (3.16)$$

- The de Broglie wave group associated with a moving body travels with the same velocity as the body.

The phase velocity v_p of de Broglie waves is, as we found earlier,

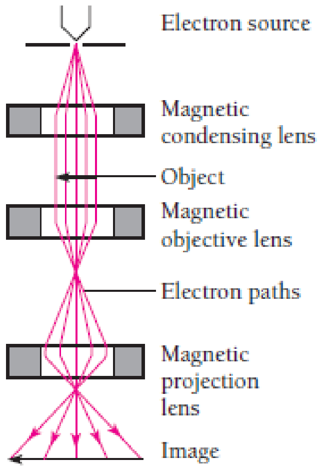
De Broglie phase velocity

$$v_p = \frac{\omega}{k} = \frac{c^2}{v} \qquad (3.3)$$

- This exceeds both the velocity of the body v and the velocity of light c, since v<c. However, v_p has no physical significance because the motion of the wave group, not the motion of the individual waves that make up the group, corresponds to the motion of the body, and $v_g < c$ as it should be.
- The fact that $v_p > c$ for de Broglie waves therefore does not violate special relativity.

Example 3.3
An electron has a de Broglie wavelength of 2.00 pm $=2.00\times10^{-12}$ m. Find its kinetic energy and the phase and group velocities of its de Broglie waves.

Electron Microscopes



- The wave nature of moving electrons is the basis of the electron microscope, the first of which was built in 1932.
- The resolving power of any optical instrument, which is limited by diffraction, is proportional to the wavelength of whatever is used to illuminate the specimen.
- In the case of a good microscope that uses visible light, the maximum useful magnification is about 500; higher magnifications give larger images but do not reveal any more detail.
- Fast electrons, however, have wavelengths very much shorter than those of visible light and are easily controlled by electric and magnetic fields because of their charge.

Figure 3.5 Because the wavelengths of the fast electrons in an electron microscope are shorter than those of the light waves in an optical microscope, the electron microscope can produce sharp images at higher magnifications.

- In an electron microscope, current-carrying coils produce magnetic fields that act as lenses to focus an electron beam on a specimen and then produce an enlarged image on a fluorescent screen or photographic plate (Fig. 3.5).
  - The technology of magnetic “lenses” does not permit the full theoretical resolution of electron waves to be realized in practice.
  - For instance, 100-keV electrons have wavelengths of 0.0037 nm, but the actual resolution they can provide in an electron microscope may be only about 0.1 nm.
- However, this is still a great improvement on the ~200-nm resolution of an optical microscope, and magnifications of over 1,000,000 have been achieved with electron microscopes.

3.5 PARTICLE DIFFRACTION

- An experiment that confirms the existence of de Broglie waves.
- A wave effect with no analog in the behavior of Newtonian particles is diffraction.
- In 1927 Clinton Davisson and Lester Germer in the United States and G. P. Thomson in England independently confirmed de Broglie’s hypothesis by demonstrating that electron beams are diffracted when they are scattered by the regular atomic arrays of crystals. (All three received Nobel Prizes for their work)
- Classical physics predicts that the scattered electrons will emerge in all directions with only a moderate dependence of
  - their intensity on scattering angle and
  - even less on the energy of the primary electrons.
- Using a block of nickel as the target, Davisson and Germer verified these predictions.
- In the middle of their work an accident occurred that allowed air to enter their apparatus and oxidize the metal surface. To reduce the oxide to pure nickel, the target was baked in a hot oven.
- After this treatment, the target was returned to the apparatus and the measurements resumed.

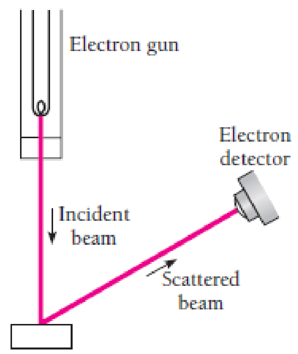


Figure 3.6 The Davisson-Germer experiment.

- Now the results were very different. Instead of a continuous variation of scattered electron intensity with angle, distinct maxima and minima were observed whose positions depended upon the electron energy!
- Typical polar graphs of electron intensity after the accident are shown in Fig. 3.7.

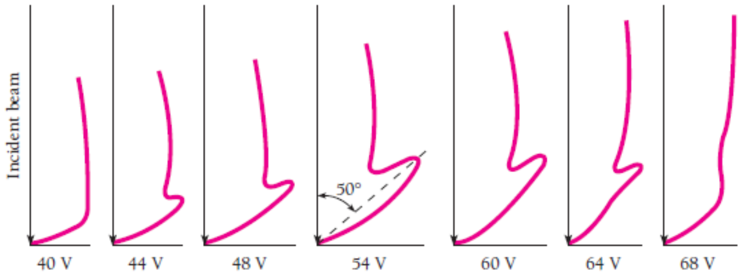


Figure 3.7 Results of the Davisson-Germer experiment, showing how the number of scattered electrons varied with the angle between the incoming beam and the crystal surface. The Bragg planes of atoms in the crystal were not parallel to the crystal surface, so the angles of incidence and scattering relative to one family of these planes were both 65° (see Fig. 3.8).

- The method of plotting is such that the intensity at any angle is proportional to the distance of the curve at that angle from the point of scattering.
- If the intensity were the same at all scattering angles, the curves would be circles (scattered electrons emerge in all directions) centered on the point of scattering.
- De Broglie's hypothesis suggested that electron waves were being diffracted by the target, much as x-rays are diffracted by planes of atoms in a crystal.
- Let us see whether we can verify that de Broglie waves are responsible for the findings of Davisson and Germer. In a particular case,
    - a beam of 54-eV electrons was directed perpendicularly at the nickel target and
    - a sharp maximum in the electron distribution occurred at an angle of 50° with the original beam.
    - The angles of incidence and scattering relative to the family of Bragg planes shown in Fig. 3.8 are both 65°.

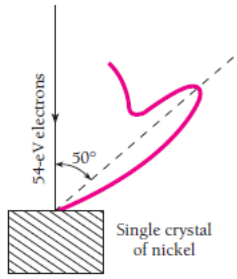


Figure 3.8 The diffraction of the de Broglie waves by the target is responsible for the results of Davisson and Germer.

- The spacing of the planes in this family, which can be measured by x-ray diffraction, is 0.091 nm. The Bragg equation for maxima in the diffraction pattern is

$$n\lambda = 2d \sin\theta \qquad (2.13)$$

Here $d = 0.091$ nm and $\theta = 65°$. For $n = 1$ the de Broglie wavelength λ of the diffracted electrons is

$$\lambda = 2d \sin\theta = (2)(0.091 \text{ nm})(\sin 65°) = 0.165 \text{ nm}$$

Now we use de Broglie's formula $\lambda = h/\gamma mv$ to find the expected wavelength of the electrons. The electron kinetic energy of 54 eV is small compared with its rest energy mc^2 of 0.51 MeV, so we can let $\gamma = 1$. Since

$$\text{KE} = \tfrac{1}{2}mv^2$$

the electron momentum mv is

$$\begin{aligned} mv &= \sqrt{2m\text{KE}} \\ &= \sqrt{(2)(9.1 \times 10^{-31}\text{ kg})(54 \text{ eV})(1.6 \times 10^{-19}\text{ J/eV})} \\ &= 4.0 \times 10^{-24}\text{ kg}\cdot\text{m/s} \end{aligned}$$

The electron wavelength is therefore

$$\lambda = \frac{h}{mv} = \frac{6.63 \times 10^{-34}\text{ J}\cdot\text{s}}{4.0 \times 10^{-24}\text{ kg}\cdot\text{m/s}} = 1.66 \times 10^{-10}\text{ m} = 0.166 \text{ nm}$$

- which agrees well with the observed wavelength of 0.165 nm. The Davisson-Germer experiment thus directly verifies de Broglie's hypothesis of the wave nature of moving bodies.

3.6 PARTICLE IN A BOX

- Why the energy of a trapped particle is quantized.
- The wave nature of a moving particle leads to some remarkable consequences when the particle is restricted to a certain region of space instead of being able to move freely.
- The simplest case is that of a particle that bounces back and forth between the walls of a box, as in Fig. 3.9.

- From a wave point of view, a particle trapped in a box is like a standing wave in a string stretched between the box's walls.
- In both cases the wave variable (transverse displacement for the string, wave function   for the moving particle) must be 0 at the walls, since the waves stop there.

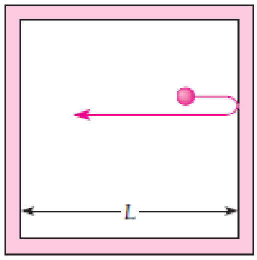


Figure 3.9 A particle confined to a box of width L. The particle is assumed to move back and forth along a straight line between the walls of the box.

- The possible de Broglie wavelengths of the particle in the box therefore are determined by the width L of the box, as in Fig. 3.10.

De Broglie wavelengths of trapped particle

$$\lambda_n = \frac{2L}{n} \qquad n = 1, 2, 3, \ldots \qquad (3.17)$$

- The kinetic energy of a particle of momentum mv is

$$\text{KE} = \tfrac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{h^2}{2m\lambda^2}$$

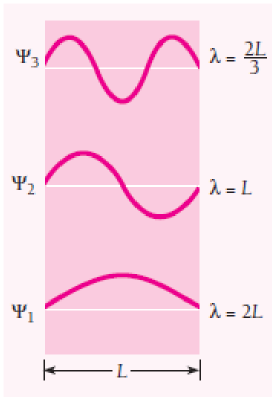


Figure 3.10 Wave functions of a particle trapped in a box L wide.

- The particle has no potential energy in this model, the only energies it can have are

Particle in a box

$$E_n = \frac{n^2h^2}{8mL^2} \qquad n = 1, 2, 3, \ldots \qquad (3.18)$$

- Each permitted energy is called an energy level, and the integer n that specifies an energy level E_n is called its **quantum number**.
- We can draw three general conclusions from Eq. (3.18). These conclusions apply to any particle confined to a certain region of space (even if the region does not have a well-defined boundary), for instance an atomic electron held captive by the attraction of the positively charged nucleus.

1. A trapped particle cannot have an arbitrary energy, as a free particle can. The fact of its confinement leads to restrictions on its wave function that allow the particle to have only certain specific energies and no others.
2. A trapped particle cannot have zero energy. Since the de Broglie wavelength of the particle is λ=h/mv, a speed of v=0 means an infinite wavelength. But there is no way to reconcile an infinite wavelength with a trapped particle, so such a particle must have at least some kinetic energy.
3. Because Planck's constant is so small-only $6.63\text{x}10^{-34}$ J.s-quantization of energy is considerable only when m and L are also small. This is why we are not aware of energy quantization in our own experience.

Example 3.4&3.5

An electron is in a box 0.10 nm across, which is the order of magnitude of atomic dimensions. Find its permitted energies.

A 10-g marble is in a box 10 cm across. Find its permitted energies.