1 Approximation of Functions

- To get the value of $sin(2.113)$ or $e^{-3.5}$.
- Does NOT look up in tables and interpolate!
- The computer approximates every function **from some polynomial** that is customized to give the values very accurately.
- We want the approximation to be efficient in that it obtains the values with the ***smallest error*** in the ***least number of arithmetic operations***.
- Another way to approximate a function is with a series of *sine* and *cosine* terms, Fourier series (represents *periodic* functions).
- **Chebyshev Polynomials and Chebyshev Series:** Chebyshev polynomials are ***orthogonal polynomials*** that are the **basis for fitting nonalgebraic functions** with maximum efficiency.
- They can be used to modify a Taylor series so that there is greater efficiency.
- A series of such polynomials converges more rapidly than a Taylor series.
- **Fourier Series:** These are series of sine and cosine terms that can be used to approximate a function within a given interval very closely, even functions with discontinuities.
- Fourier series are important in many areas, particularly in getting an analytical solution to partial-differential equations.
- If we want to represent a known function as a polynomial, one way to do it is with a **Taylor series**.
- Given a function, $f(x)$, we write

$$P_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \ldots + a_n(x-a)^n + \ldots$$

- Where

$$a_i = \frac{f^{(i)}(a)}{i!}$$

- Then, rewriting this Taylor series expansion as

$$P_n(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$
$$+\frac{f^n(a)}{n!}(x-a)^n + \dots$$

- Unless $f(x)$ is itself a polynomial, the series may have an **infinite number of terms**.
- Terminating the series incurs an error, **truncation error**.
- The error after the $(x-a)^n$ term,

$$Error = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), where\ \xi\ in\ [a, x]$$

- A problem with using the Taylor series to get polynomial approximations to a transcendental function is that the error grows rapidly as x-values <u>*depart from*</u> $x = a$.
- For $f(x) = e^x$, the Taylor series is easy to write because the derivatives are so simple: $f^n(a) = e^a$ for all orders (n),
- For $a = 0$, we have, (which is then called a *Maclaurin series*)

$$e^x \approx 1 + 1(x-0) + 1/2(x-0)^2 + 1/6(x-0)^3$$

- if we use only terms through x^3; the error term shows that the error of this will grow about proportional to x^4 as x-values ***depart from zero***.
- There is a way to deal with this <u>rapid growth of the errors</u>,
- That is to write the polynomial approximation to $f(x)$ in terms of *Chebyshev polynomials*.

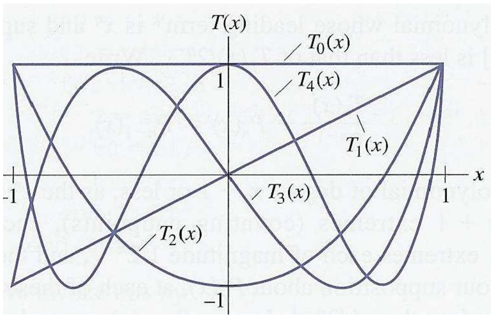

Figure 1: Plot of the first four polynomials of the Chebyshev polynomials.

1.1 Chebyshev Polynomials and Chebyshev Series

1.1.1 Chebyshev Polynomials

- A Maclaurin series can be thought of as representing $f(x)$ as a weighted sum of polynomials.

- The kind of polynomials that are used are just the successive powers of x: $1, x, x^2, x^3, \ldots$.

- Chebyshev polynomials are not as simple; the first 11 of these are

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_3(x) &= 4x^3 - 3x \\
T_4(x) &= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \\
T_8(x) &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \\
T_9(x) &= 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x \\
T_{10}(x) &= 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1
\end{aligned} \tag{1}$$

- Note that the coefficient of x^n in $T_n(x)$ is always 2^{n-1}.

- In Fig. 1, we plot the first four polynomials of Eqn.1.

- The members of this series of polynomials can be generated from the two-term recursion formula

$$\begin{aligned}
T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \\
T_0(x) &= 1 \quad \& \quad T_1(x) = x
\end{aligned}$$

- They form an orthogonal set,

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = \left\{ \begin{array}{ll} 0, & n \neq m \\ \pi, & n = m = 0 \\ \pi/2, & n = m \neq 0 \end{array} \right\}$$

- The Chebyshev polynomials are also terms of a Fourier series, because

$$T_n(x) = cos(n\theta)$$

where $\theta = arc(cosx)$. Observe that

$$\begin{array}{lrl} n = 0; & cos0 = 1 \rightarrow & T_0 = 1 \\ n = 1; & cos\theta = cos(arc(cosx)) = x \rightarrow & T_1 = x \end{array}$$

- Because of the relation $T_n(x) = cos(n\theta)$, the Chebyshev polynomials will have a succession of maxima and minima of alternating signs, as Figure 1 shows.

- MATLAB has no commands for these polynomials but this M-file will compute them:

```
function T=Tch(n)
if n==0
        disp('1')
elseif n==1
        disp('x')
else
        t0='1';
        t1='x';
        for i=2:n
                T=symop('2*x','*',t1,'-',t0);
                t0=t1;
                t1=T;
        end
end
>>Tch(5)
>>collect(ans)
ans= 16*x^5-20*x^3+5*x
```

if *symop* does not exist, download.

- All polynomials of degree n that have a coefficient of one on x^n, the polynomial

$$\frac{1}{2^{n-1}} T_n(x)$$

has a smaller upper bound to its magnitude in the interval [-1, 1].

- This is important because we will be able to write power function approximations to functions whose maximum errors are given in terms of this upper bound.

- **Example m-file:** Comparison of Lagrangian interpolation polynomials for equidistant and non-equidistant (Chebyshev) sample points for the function $f(x) = \frac{1}{1+x^2}$ (lagrange_chebyshev.m)

1.1.2 Economizing a Power Series

- We begin a search for better power series representations of functions by using Chebyshev polynomials to *economize* a Maclaurin series.

- This will give a modification of the Maclaurin series that produces a fifth-degree polynomial

- whose errors are only slightly greater than those of a sixth-degree Maclaurin series.

- We start with a Maclaurin series for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \dots$$

- If we would like to use a truncated series to approximate e^x on the interval $[0, 1]$ with a precision of 0.001,

- We will have to retain terms through that in x^6, because the error after the term in x^5 will be more than

$$1/720 = 0.00139 \ (and\ 1/120 = 0.0084)$$

- Suppose we subtract

$$\left(\frac{1}{720}\right)\left(\frac{T_6}{32}\right)$$

from the truncated series.

- This will exactly cancel the x^6 term from Eqn. 1

- and at the same time make adjustments in other coefficients of the Maclaurin series.

- Because the maximum value of T_6 on the interval $[0, 1]$ is unity,

x	e^x	Maclaurin of degree			Economized of degree	
		6	5	4	5	4
0.0	1.00000	1.00000	1.00000	1.00000	1.00004	1.00004
0.2	1.22140	1.22140	1.22140	1.22140	1.22142	1.22098
0.4	1.49182	1.49182	1.49182	1.49173	1.49178	1.49133
0.6	1.82212	1.82212	1.82205	1.82140	1.82208	1.82212
0.8	2.22554	2.22549	2.32513	2.22240	2.22553	2.22605
1.0	2.71828	2.71806	2.71667	2.70833	2.71801	2.71749
Maximum error		0.00023	0.00162	0.00995	0.00027	0.00078

Table 1: Comparison of economized series with Maclaurin series.

- this will change the sum of the truncated series by only

$$\left(\frac{1}{720}\right)\left(\frac{1}{32}\right) < 0.00005$$

which is small with respect to our required precision of 0.001.

- Performing the calculations, we have

$$e^x \approx \overbrace{1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720}}^{Maclaurin}$$
$$- \left(\tfrac{1}{720}\right) \underbrace{\left(\frac{(32x^6 - 48x^4 + 18x^2 - 1)}{32}\right)}_{Chebyshev\ T_6(x)/32}$$

$$e^x \approx 1.000043 + x + 0.499219x^2 + \tfrac{x^3}{6} + 0.043750x^4 + \tfrac{x^5}{120}$$

- The resulting fifth-degree polynomial approximates e^x on $[0, 1]$ nearly as well as the sixth-degree Maclaurin series.
- its maximum error (at $x = 1$) is 0.000270, compared to 0.000226 for the Maclaurin polynomial (see Table 1).
- We <u>*economize*</u> in that we get about the same precision with a lower-degree polynomial.
- By subtracting $\frac{1}{120}\frac{T_5}{16}$ we can economize further, getting a fourth-degree polynomial that is almost as good as the economized fifth-degree one.

- So that we have found a fourth-degree power series that meets an error criterion that requires us to use two additional terms of the original Maclaurin series.

- Because of the relative ease with which they can be developed, such economized power series are *frequently used for approximations to functions.*

- Much more efficient than power series of the same degree obtained by truncating a Taylor or Maclaurin series.

- Observe that even the economized polynomial of degree-4 is more accurate than a fifth-degree Maclaurin series.

- Also notice that near $x = 0$, the economized polynomials are less accurate.

- We can get the economized series with MATLAB by employing our M-file for the Chebyshev series.

- We must start with x as a symbolic variable, then get the Maclaurin series and subtract the proper multiple of the Chebyshev series:

```
>> syms x
>> ts=taylor(exp(x),7)
1+x+1/2*x^2+1/6*x^3+1/24*x^4+1/120*x^5+1/720*x^6
>> cs=Tch(6);
>> es=ts-cs/factorial(6)/2^5
es=23041/23040+x+639/1280*x^2+1/6*x^3+7/160*x^4+1/120*x^5
>> vpa(es,7)
>> collect(ans)
```

1.1.3 Chebyshev Series

- By rearranging the Chebyshev polynomials,
- we can express powers of x in terms of them:

$$\begin{aligned} 1 &= T_0 \\ x &= T_1 \\ x^2 &= \tfrac{1}{2}(T_0 + T_2) \\ x^3 &= \tfrac{1}{4}(3T_1 + T_3) \\ x^4 &= \tfrac{1}{8}(3T_0 + 4T_2 + T_4) \\ x^5 &= \tfrac{1}{16}(10T_1 + 5T_3 + T_5) \\ x^6 &= \tfrac{1}{32}(10T_0 + 15T_2 + 6T_4 + T_6) \\ x^7 &= \tfrac{1}{64}(35T_1 + 21T_3 + 7T_5 + T_7) \\ x^8 &= \tfrac{1}{128}(35T_0 + 56T_2 + 28T_4 + 8T_6 + T_8) \\ x^9 &= \tfrac{1}{256}(126T_1 + 84T_3 + 36T_5 + 9T_7 + T_9) \end{aligned} \quad (2)$$

- By substituting these identities into an infinite Taylor series
- and collecting terms in $T_i(x)$, we create a Chebyshev series.
- For example, we can get the first four terms of a Chebyshev series

$$e^x = 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3$$

  by starting with the Maclaurin expansion for e^x.

- Such a series **converges more rapidly** than does a Taylor series on $[-1, 1]$;

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$$

- Replacing terms by Eqn. 2, but omitting polynomials beyond $T_3(x)$ because we want only four terms, we have;

$$e^x = 1.2661T_0 + 1.1302T_1 + 0.2715T_2 + 0.0443T_3 + \dots$$

- The number of terms that are employed determines the accuracy of the computed values.
- To compare the Chebyshev expansion with the Maclaurin series, we convert back to powers of x, using Eqn. 1:

$$e^x = 0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3 + \dots \quad (3)$$

x	e^x	Chebyshev	Error	Maclaurin	Error
−1.0	0.3679	0.3631	0.0048	0.3333	0.0346
−0.8	0.4493	0.4536	−0.0042	0.4346	0.0147
−0.6	0.5488	0.5534	−0.0046	0.5440	0.0048
−0.4	0.6703	0.6712	−0.0009	0.6693	0.0010
−0.2	0.8187	0.8154	0.0033	0.8187	0.0001
0	1.0000	0.9946	0.0054	1.0000	0.0000
0.2	1.2214	1.2172	0.0042	1.2213	0.0001
0.4	1.4918	1.4917	0.0001	1.4907	0.0012
0.6	1.8221	1.8267	−0.0046	1.8160	0.0061
0.8	2.2255	2.2307	−0.0051	2.2054	0.0202
1.0	2.7183	2.7121	0.0062	2.6667	0.0516

Table 2: Comparison of Chebyshev series for e^x with Maclaurin series.

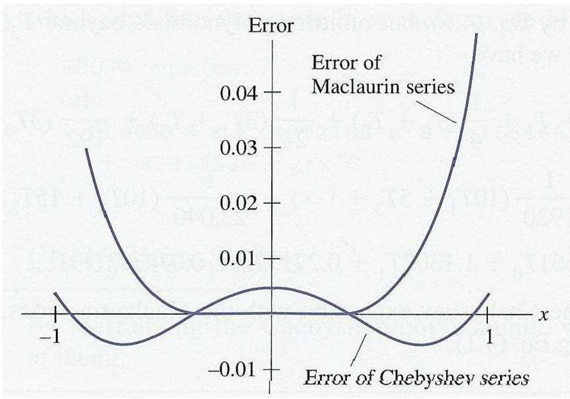

Figure 2: Comparison of the error of Chebyshev series for e^x with the error of Maclaurin series.

- Table 2 and Figure 2 compare the error of the Chebyshev expansion $(0.9946 + 0.9973x + 0.5430x^2 + 0.1772x^3)$ with the Maclaurin series $(1 + x + 0.5x^2 + 0.1667x^3)$.
  - Chebyshev expansion, the errors can be considered to be distributed more or less **uniformly throughout the interval**.
  - Maclaurin expansion, which gives very small errors near the origin, allows the error to bunch up at the ends of the interval.