Lecture 6

Mathematical Expectation

Lecture Information

Ceng272 *Statistical Computations* at March 22, 2010

Dr. Cem Özdoğan
Computer Engineering Department
Çankaya University

Contents

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.
- For a measure of central tendency of the random variable X we use the terms **expectation**, **expected value**, and **average value** for the same concept.

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.
- For a measure of central tendency of the random variable X we use the terms **expectation**, **expected value**, and **average value** for the same concept.
- Intuitively, the expected value of X is the average value that the random variable takes on.

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.
- For a measure of central tendency of the random variable X we use the terms **expectation**, **expected value**, and **average value** for the same concept.
- Intuitively, the expected value of X is the average value that the random variable takes on.
- However, some of the values of the random variable X could be more (or less) probable than the other in the distribution unless the random variable is distributed uniformly.

Mean of a Random Variable I

- Suppose that a probability distribution of a random variable X is specified.
- For a measure of central tendency of the random variable X we use the terms **expectation**, **expected value**, and **average value** for the same concept.
- Intuitively, the expected value of X is the average value that the random variable takes on.
- However, some of the values of the random variable X could be more (or less) probable than the other in the distribution unless the random variable is distributed uniformly.
- Hence, in order to consider an **average** value of X we need to take its probability into account.

Mean of a Random Variable II

- If I repeat the experiment many times, what would be the average number of an outcome of a random variable?

Mean of a Random Variable II

- If I repeat the experiment many times, what would be the average number of an outcome of a random variable?
- **Definition 4.1**:

  Let X be a random variable with probability distribution $f(x)$. The **mean** or **expected values** of X is

$$\left\{ \begin{array}{l} \mu = E(X) = \sum_x xf(x) \textit{ if } X \textit{ is discrete} \\ \mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \textit{ if } X \textit{ is continuous} \end{array} \right\}$$

Mean of a Random Variable II

- If I repeat the experiment many times, what would be the average number of an outcome of a random variable?
- **Definition 4.1**:

Let X be a random variable with probability distribution $f(x)$. The **mean** or **expected values** of X is

$$\left\{ \begin{array}{l} \mu = E(X) = \sum_x x f(x) \textit{ if } X \textit{ is discrete} \\ \mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx \textit{ if } X \textit{ is continuous} \end{array} \right\}$$

- The expected value is used <u>as a measure of centering or location</u> of the distribution of a random variable X.

Mean of a Random Variable II

- If I repeat the experiment many times, what would be the average number of an outcome of a random variable?
- **Definition 4.1**:

  Let X be a random variable with probability distribution $f(x)$. The **mean** or **expected values** of X is

  $$\left\{ \begin{array}{l} \mu = E(X) = \sum_x xf(x) \text{ if } X \text{ is discrete} \\ \mu = E(X) = \int_{-\infty}^{\infty} xf(x)dx \text{ if } X \text{ is continuous} \end{array} \right\}$$

- The expected value is used as a measure of centering or location of the distribution of a random variable X.
- By the uniform distribution assumption, i.e. all values of X are equally likely to occur in population with size N, $f(x) = \frac{1}{N}$ for all x,

$$E(X) = \sum_x xf(x) = \sum_x x(\frac{1}{N}) = (\frac{1}{N})\sum_i x_i = \mu = \bar{x}$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.
- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.
- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.
- The average number of heads per toss is then

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.
- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.
- The average number of heads per toss is then

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.
- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.
- The average number of heads per toss is then

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16}$$

where $\frac{4}{16}, \frac{7}{16}, \frac{5}{16}$ are relative frequencies

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable III

- **Example**: If two coins are tossed 16 times and X is the number of heads that occur per toss, then the value of X can be 0, 1, 2.
- The experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively.
- The average number of heads per toss is then

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16}$$

x	0	1	2
$f(x)$	4/16	7/16	5/16

where $\frac{4}{16}, \frac{7}{16}, \frac{5}{16}$ are relative frequencies

$$0 * \frac{4}{16} + 1 * \frac{7}{16} + 2 * \frac{5}{16} = \frac{17}{16} = 1.0625$$

Mean of a Random Variable IV

- **Example 4.1**: A lot contain 4 good components and 3 defective components.

Mean of a Random Variable IV

- **Example 4.1**: A lot contain 4 good components and 3 defective components.
  - A sample of 3 is taken by a quality inspector.

Mean of a Random Variable IV

- **Example 4.1**: A lot contain 4 good components and 3 defective components.
  - A sample of 3 is taken by a quality inspector.
  - Find the expected value of the number of good components in this sample.

Mean of a Random Variable IV

- **Example 4.1**: A lot contain 4 good components and 3 defective components.
  - A sample of 3 is taken by a quality inspector.
  - Find the expected value of the number of good components in this sample.
- Solution: X represents the number of good components

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, x = 0, 1, 2, 3$$

$$\mu = E(X) = 0 * f(0) + 1 * f(1) + 2 * f(2) + 3 * f(3) = \frac{12}{7}$$

Mean of a Random Variable V

- **Example 4.3**: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is as the following.

$$f(x) = \left\{ \begin{array}{ll} \frac{20000}{x^3}, & x > 100 \\ 0, & elsewhere \end{array} \right\}$$

Find the expected life of this type of device.

Mean of a Random Variable V

- **Example 4.3**: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is as the following.

$$f(x) = \left\{ \begin{array}{ll} \frac{20000}{x^3}, & x > 100 \\ 0, & elsewhere \end{array} \right\}$$

Find the expected life of this type of device.

- Solution:

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20000}{x^3} dx = -\frac{20000}{x} |_{100}^{\infty} = 200$$

Mean of a Random Variable VI

- **Mean of** $g(X)$ (any real-valued function): If X is a discrete random variable with $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$ then

$$
\begin{aligned}
P[g(X) = 0] &= P(X = 0) = f(0),\\
P[g(X) = 1] &= P(X = -1) + P(X = 1) = f(-1) + f(1),\\
P[g(X) = 4] &= P(X = 2) = f(2),
\end{aligned}
$$

Mean of a Random Variable VI

- **Mean of** $g(X)$ (any real-valued function): If X is a discrete random variable with $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$ then

$$P[g(X) = 0] = P(X = 0) = f(0),$$
$$P[g(X) = 1] = P(X = -1) + P(X = 1) = f(-1) + f(1),$$
$$P[g(X) = 4] = P(X = 2) = f(2),$$

- The probability distribution of $g(X)$ can be written

$g(x)$	0	1	4
$P[g(X) = 4]$	f(0)	f(-1)+f(1)	f(2)

Mean of a Random Variable VI

- **Mean of** $g(X)$ (any real-valued function): If X is a discrete random variable with $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$ then

$$\begin{aligned} P[g(X) = 0] &= P(X = 0) = f(0), \\ P[g(X) = 1] &= P(X = -1) + P(X = 1) = f(-1) + f(1), \\ P[g(X) = 4] &= P(X = 2) = f(2), \end{aligned}$$

- The probability distribution of $g(X)$ can be written

$g(x)$	0	1	4
$P[g(X) = 4]$	f(0)	f(-1)+f(1)	f(2)

-

$$\begin{aligned} E(g(X)) &= 0 * f(0) + 1 * [f(-1) + f(1)] + 4 * f(2) \\ &= (-1)^2 * f(-1) + (0)^2 * f(0) + (1)^2 * f(1) + (2)^2 * f(2) \\ &= \sum_x g(x) * f(x) \end{aligned}$$

Mean of a Random Variable VII

- **Theorem 4.1::**

  Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

$$\begin{cases} \mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x) \text{ if } X \text{ is discrete} \\ \mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \text{ if } X \text{ is continuous} \end{cases}$$

Mean of a Random Variable VII

- **Theorem 4.1:**:

  Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

  $$\left\{ \begin{array}{l} \mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x) \textit{ if } X \textit{ is discrete} \\ \mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \textit{ if } X \textit{ is continuous} \end{array} \right\}$$

- **Example 4.5**: Let X be a random variable with density function

  $$f(x) = \left\{ \begin{array}{ll} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & \textit{elsewhere} \end{array} \right\}$$

Mean of a Random Variable VII

- **Theorem 4.1:**:

  Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

  $$\left\{ \begin{array}{l} \mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x) \textit{ if } X \textit{ is discrete} \\ \mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \textit{ if } X \textit{ is continuous} \end{array} \right\}$$

- **Example 4.5**: Let X be a random variable with density function

  $$f(x) = \left\{ \begin{array}{cc} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{array} \right\}$$

- Find the expected value of $g(X) = 4X + 3$.

Mean of a Random Variable VII

- **Theorem 4.1:**:

  Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

  $$\left\{ \begin{array}{l} \mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x) \textit{ if } X \textit{ is discrete} \\ \mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx \textit{ if } X \textit{ is continuous} \end{array} \right\}$$

- **Example 4.5**: Let X be a random variable with density function

  $$f(x) = \left\{ \begin{array}{ll} \frac{x^2}{3}, & -1 < x < 2 \\ 0, & elsewhere \end{array} \right\}$$

- Find the expected value of $g(X) = 4X + 3$.
- Solution:

$$E[g(X)] = E(4X + 3) = \frac{1}{3}\int_{-1}^{2}(4x^3 + 3x^2)dx = 8$$

Mean of a Random Variable VIII

- **Theorem 4.2:**:

  Let X and Y be random variables with joint probability function $f(x, y)$. The mean of the random variable $g(X, Y)$ is

$$\left\{ \begin{array}{l} \mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y) f(x, y) \\ \text{if } X \text{ and } Y \text{ are discrete} \\ \mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dxdy \\ \text{if } X \text{ and } Y \text{ are continuous} \end{array} \right\}$$

Mean of a Random Variable VIII

- **Theorem 4.2:**:

  Let X and Y be random variables with joint probability function $f(x, y)$. The mean of the random variable $g(X, Y)$ is

$$\left\{ \begin{array}{l} \mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y) \\ \text{if } X \text{ and } Y \text{ are discrete} \\ \mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dxdy \\ \text{if } X \text{ and } Y \text{ are continuous} \end{array} \right\}$$

- **Example 4.7**: Find $E(Y/X)$ for the density function

$$f(x,y) = \left\{ \begin{array}{ll} \frac{x(1+3y^2)}{4}, & 0 < x < 2, \; , 0 < y < 1 \\ 0, & elsewhere \end{array} \right\}$$

Mean of a Random Variable VIII

- **Theorem 4.2:**:

Let X and Y be random variables with joint probability function $f(x,y)$. The mean of the random variable $g(X,Y)$ is

$$\left\{ \begin{array}{l} \mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y) f(x,y) \\ \textit{if } X \textit{ and } Y \textit{ are discrete} \\ \mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dxdy \\ \textit{ if } X \textit{ and } Y \textit{ are continuous} \end{array} \right\}$$

- **Example 4.7**: Find $E(Y/X)$ for the density function

$$f(x,y) = \left\{ \begin{array}{ll} \frac{x(1+3y^2)}{4}, & 0 < x < 2, \ , 0 < y < 1 \\ 0, & \textit{elsewhere} \end{array} \right\}$$

- Solution:

$$E(\frac{Y}{X}) = \int_0^1 \int_0^2 \frac{y}{x} \frac{x(1+3y^2)}{4} dxdy = \frac{5}{8}$$

Mean of a Random Variable IX

- If $g(X,Y) = X$ is

$$E(X) = \left\{ \begin{array}{l} \sum_x \sum_y x f(x,y) = \sum_x x g(x) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\,dx\,dy = \int_{-\infty}^{\infty} x g(x)\,dx \end{array} \right\}$$

where $g(x)$ is the marginal distribution of X

Mean of a Random Variable IX

- If $g(X,Y) = X$ is

$$E(X) = \left\{ \begin{array}{l} \sum_x \sum_y xf(x,y) = \sum_x xg(x) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x,y)dxdy = \int_{-\infty}^{\infty} xg(x)dx \end{array} \right\}$$

where $g(x)$ is the marginal distribution of X

- If $g(X,Y) = Y$ is

$$E(Y) = \left\{ \begin{array}{l} \sum_x \sum_y yf(x,y) = \sum_y yh(y) \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x,y)dxdy = \int_{-\infty}^{\infty} yh(y)dy \end{array} \right\}$$

where $h(y)$ is the marginal distribution of Y

Variance and Covariance I

- A mean does not give adequate description of the shape of a random variable (probability distribution).

Variance and Covariance I

- A mean does not give adequate description of the shape of a random variable (probability distribution).
- We need to characterize the variability (or dispersion) of the random variable X in the distribution.

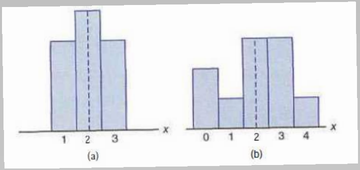

Figure: Distributions with equal means and unequal dispersions.

Variance and Covariance I

- A mean does not give adequate description of the shape of a random variable (probability distribution).
- We need to characterize the variability (or dispersion) of the random variable X in the distribution.

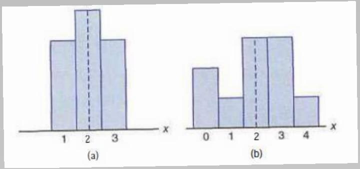

Figure: Distributions with equal means and unequal dispersions.

- **Definition 4.3**:

Let X be a random variable with probability distribution $f(x)$ and mean μ. The **variance** of X is

$$\left\{ \begin{array}{l} \sigma^2 = E\left[(X-\mu)^2\right] = \sum_x (x-\mu)^2 f(x), \text{ if } X \text{ is discrete} \\ \sigma^2 = E\left[(X-\mu)^2\right] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \text{ if } X \text{ is continuous} \end{array} \right\}$$

σ is called the **standard deviation** of X.

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.

Variance and Covariance II

- **Example 4.8**:Let the random variable *X* represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

x	1	2	3
$f(x)$	0.3	0.4	0.3

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

x	1	2	3
$f(x)$	0.3	0.4	0.3

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

x	1	2	3
$f(x)$	0.3	0.4	0.3

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

- Show that the variance of the probability distribution for company B is greater than that of company A.

Variance and Covariance II

- **Example 4.8**:Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday.
- The probability distribution for company A and B is as follows.

x	1	2	3
$f(x)$	0.3	0.4	0.3

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

- Show that the variance of the probability distribution for company B is greater than that of company A.
- Solution:

$$\mu_A = E(X) = 1 * 0.3 + 2 * 0.4 + 3 * 0.3 = 2.0$$

$$\sigma_A^2 = \sum_{x=1}^{3} (x - 2.0)^2 f(x) = (1-2)^2 * 0.3 + (2-2)^2 * 0.4 + (3-2)^2 * 0.3 = 0.6$$

$$\mu_B = 2.0 \ \& \ \sigma_B^2 = 1.6$$

Variance and Covariance III

- **Theorem 4.2**:

  The **variance** of a random variable X is

  $$\sigma^2 = E(X^2) - \mu^2$$

Variance and Covariance III

- **Theorem 4.2**:

  The **variance** of a random variable X is

  $$\sigma^2 = E(X^2) - \mu^2$$

- **Example 4.9**: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested.

Variance and Covariance III

- **Theorem 4.2**:

> The **variance** of a random variable X is
>
> $$\sigma^2 = E(X^2) - \mu^2$$

- **Example 4.9**: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested.
- Calculate σ^2 using the following probability distribution.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Variance and Covariance III

- **Theorem 4.2**:

  The **variance** of a random variable X is

  $$\sigma^2 = E(X^2) - \mu^2$$

- **Example 4.9**: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested.
- Calculate σ^2 using the following probability distribution.

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

- Solution:

$$\mu = E(X) = 0 * 0.51 + \ldots = 0.61$$

$$E(X^2) = \sum_{x=0}^{3} x^2 f(x) = 0^2 * 0.51 + \ldots = 0.87$$

$$\sigma^2 = E(X^2) - \mu^2 = 0.87 - 0.61^2 = 0.4979$$

Variance and Covariance IV

- **Theorem 4.3**:

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\left\{ \begin{array}{l} \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \sum_x [g(X) - \mu_{g(X)}]^2, \\ \quad \textit{if } X \textit{ is discrete} \\ \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \int_{-\infty}^{\infty} [g(X) - \mu_{g(X)}]^2 f(x)dx, \\ \quad \textit{if } X \textit{ is continuous} \end{array} \right\}$$

Variance and Covariance IV

- **Theorem 4.3**:

  Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\left\{ \begin{array}{l} \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \sum_x [g(X) - \mu_{g(X)}]^2, \\ \quad \textit{if } X \textit{ is discrete} \\ \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \int_{-\infty}^{\infty} [g(X) - \mu_{g(X)}]^2 f(x)dx, \\ \quad \textit{if } X \textit{ is continuous} \end{array} \right\}$$

- **Example 4.11**: Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution.

x	0	1	2	3
$f(x)$	1/4	1/8	1/2	1/8

Variance and Covariance IV

- **Theorem 4.3**:

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\left\{ \begin{array}{l} \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \sum_x [g(X) - \mu_{g(X)}]^2, \\ \quad \textit{if } X \textit{ is discrete} \\ \sigma^2_{g(X)} = E\left\{[g(X) - \mu_{g(X)}]^2\right\} = \int_{-\infty}^{\infty} [g(X) - \mu_{g(X)}]^2 f(x)dx, \\ \quad \textit{if } X \textit{ is continuous} \end{array} \right\}$$

- **Example 4.11**: Calculate the variance of $g(X) = 2X + 3$, where X is a random variable with probability distribution.

x	0	1	2	3
$f(x)$	1/4	1/8	1/2	1/8

- Solution:

$$\mu_{2X+3} = E(2X+3) = \sum_{x=0}^{3}(2X+3)f(x) = 6$$

$$\sigma^2_{2X+3} = E\left\{[2X+3-\mu_{2X+3}]^2\right\} = E\left\{[2X+3-6]^2\right\}$$

$$= E(4X^2 - 12X + 9) = \sum_{x=0}^{3}(4X^2 - 12X + 9)f(x) = 4$$

Variance and Covariance V

- **Definition 4.4**:

Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

$$\begin{cases} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y)f(x,y), \\ \text{if } X \text{ and } Y \text{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dxdy, \text{ if } X \text{ and} Y \text{ are continuous} \end{cases}$$

Variance and Covariance V

- **Definition 4.4**:

  Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

$$\left\{ \begin{array}{l} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y), \\ \textit{ if } X \textit{ and } Y \textit{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y)\,dxdy, \textit{ if } X \textit{ and} Y \textit{ are continuous} \end{array} \right\}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.

Variance and Covariance V

- **Definition 4.4**:

  Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

  $$\left\{ \begin{array}{l} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y), \\ \textit{ if } X \textit{ and } Y \textit{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y) dxdy, \textit{ if } X \textit{ and} Y \textit{ are continuous} \end{array} \right\}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.
- The **sign** of the covariance indicates whether the relationship between two dependent random variables is positive or negative.

Variance and Covariance V

- **Definition 4.4**:

  Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

  $$\left\{ \begin{array}{l} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y) f(x,y), \\ \text{ if } X \text{ and } Y \text{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_X)(y-\mu_Y) f(x,y) dxdy, \text{ if } X \text{ and} Y \text{ are continuous} \end{array} \right\}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.
- The **sign** of the covariance indicates whether the relationship between two dependent random variables is positive or negative.
- When X and Y are statistically independent, it can be shown that the covariance is zero.

Variance and Covariance V

- **Definition 4.4**:

  Let X and Y be random variables with joint probability distribution $f(x,y)$. The covariance of X and Y is

  $$\left\{ \begin{array}{l} \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \sum_x \sum_y (x-\mu_X)(y-\mu_Y)f(x,y), \\ \text{ if } X \text{ and } Y \text{ are discrete} \\ \sigma_{XY} = E[(X-\mu_X)(Y-\mu_Y)] = \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f(x,y)dxdy, \text{ if } X \text{ and} Y \text{ are continuous} \end{array} \right\}$$

- The covariance between two random variables is a measurement of the nature of the **association** between the two.
- The **sign** of the covariance indicates whether the relationship between two dependent random variables is positive or negative.
- When X and Y are statistically independent, it can be shown that the covariance is zero.
- The converse, however, is not generally true. Two variables may have zero covariance and still not be statistically independent.

Variance and Covariance VI

- The covariance only describe the linear relationship between two random variables.

Variance and Covariance VI

- The covariance only describe the linear relationship between two random variables.
- If a covariance between X and Y is zero, X and Y may have a nonlinear relationship, which means that they are not necessarily independent.

Variance and Covariance VI

- The covariance only describe the linear relationship between two random variables.
- If a covariance between X and Y is zero, X and Y may have a nonlinear relationship, which means that they are not necessarily independent.
- **Theorem 4.4**:

  The covariance of two random variables X and Y with means μ_X and μ_Y respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y$$

Variance and Covariance VII

- **Definition 4.5**:

  Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The **correlation coefficient** of X and Y is

  $$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, \quad -1 \leq \rho_{XY} \leq 1$$

Variance and Covariance VII

- **Definition 4.5**:

  Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y. The **correlation coefficient** of X and Y is

  $$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}, \quad -1 \leq \rho_{XY} \leq 1$$

- Exact linear dependency: $Y = a + bX$

$$\rho_{XY} = 1, \ if \ b > 0 \ ; \ \rho_{XY} = -1, \ if \ b < 0$$

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.
- These properties will permit us to deal with expectations in terms of other parameters that are either known or are easily computed.

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.
- These properties will permit us to deal with expectations in terms of other parameters that are either known or are easily computed.
- **Theorem 4.5**:

  If a and b are constants, then

$$E(aX + b) = aE(X) + b$$

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.
- These properties will permit us to deal with expectations in terms of other parameters that are either known or are easily computed.
- **Theorem 4.5**:

  If a and b are constants, then

  $$E(aX + b) = aE(X) + b$$

- Corollary 4.1: $E(b) = b$

Means and Variance of Linear Combinations of Random Variables I

- Some useful properties that will simplify the calculations of means and variances of random variables.
- These properties will permit us to deal with expectations in terms of other parameters that are either known or are easily computed.
- **Theorem 4.5**:

  If a and b are constants, then

  $$E(aX + b) = aE(X) + b$$

- Corollary 4.1: $E(b) = b$
- Corollary 4.2: $E(aX) = aE(X)$

Means and Variance of Linear Combinations of Random Variables II

- **Example 4.16**: Applying Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$, the density function of X is as follows.

$$f(x) = \left\{ \begin{array}{ll} \frac{x^2}{3} & for \quad -1 < x < 2 \\ 0, & elsewhere \end{array} \right\}$$

Means and Variance of Linear Combinations of Random Variables II

- **Example 4.16**: Applying Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$, the density function of X is as follows.

$$f(x) = \left\{ \begin{array}{ll} \frac{x^2}{3} & for \quad -1 < x < 2 \\ 0, & elsewhere \end{array} \right\}$$

- Solution:

$$E(4X + 3) = 4E(X) + 3 = 4\left(\int_{-1}^{2} x\frac{x^2}{3}dx\right) + 3 = 8$$

Means and Variance of Linear Combinations of Random Variables II

- **Example 4.16**: Applying Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$, the density function of X is as follows.

$$f(x) = \begin{cases} \frac{x^2}{3} \text{ for } & -1 < x < 2 \\ 0, & elsewhere \end{cases}$$

- Solution:

$$E(4X + 3) = 4E(X) + 3 = 4\left(\int_{-1}^{2} x\frac{x^2}{3}dx\right) + 3 = 8$$

- **Theorem 4.6**:

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)]$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$.

$$E[X \pm Y] = E(X) \pm E(Y)$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$.

$$E[X \pm Y] = E(X) \pm E(Y)$$

- **Theorem 4.7**:

Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y)$$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$.

$$E[X \pm Y] = E(X) \pm E(Y)$$

- **Theorem 4.7**:

Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y)$$

- Corollary 4.5: Let X and Y be two independent random variables, Then $\sigma_{XY} = 0$

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$.

$$E[X \pm Y] = E(X) \pm E(Y)$$

- **Theorem 4.7**:

Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y)$$

- Corollary 4.5: Let X and Y be two independent random variables, Then $\sigma_{XY} = 0$
  - $E(XY) = E(X)E(Y)$ for independent variables

Means and Variance of Linear Combinations of Random Variables III

- **Theorem 4.7**:

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)]$$

- Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$.

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)]$$

- Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$.

$$E[X \pm Y] = E(X) \pm E(Y)$$

- **Theorem 4.7**:

Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y)$$

- Corollary 4.5: Let X and Y be two independent random variables, Then $\sigma_{XY} = 0$
  - $E(XY) = E(X)E(Y)$ for independent variables
  - $\sigma_{XY} = E(XY) - E(X)E(Y) = 0$

Means and Variance of Linear Combinations of Random Variables IV

- **Example 4.19**: In producing gallium-arsenide microchips, it is known that the ratio between gallium and arsenide is independent of producing a high percentage of workable wafers.

Means and Variance of Linear Combinations of Random Variables IV

- **Example 4.19**: In producing gallium-arsenide microchips, it is known that the ratio between gallium and arsenide is independent of producing a high percentage of workable wafers.
- Let X denote the ratio of gallium to arsenide and Y denote the percentage of workable wafers retrieved during a 1-hour period.

Means and Variance of Linear Combinations of Random Variables IV

- **Example 4.19**: In producing gallium-arsenide microchips, it is known that the ratio between gallium and arsenide is independent of producing a high percentage of workable wafers.
- Let X denote the ratio of gallium to arsenide and Y denote the percentage of workable wafers retrieved during a 1-hour period.
- X and Y are independent random variables with the joint density being known as

$$f(x) = \left\{ \begin{array}{l} \frac{x(1+3y^2)}{4} \; for \; 0 < x < 2, \;\; 0 < y < 1 \\ 0, \; elsewhere \end{array} \right\}$$

Illustrate that $E(XY) = E(X)E(Y)$.

Means and Variance of Linear Combinations of Random Variables V

- Solution:

$$E(XY) = \int_0^1 \int_0^2 xyf(x,y)dxdy = \int_0^1 \int_0^2 xy\frac{x(1+3y^2)}{4}dxdy = \frac{5}{6}$$

$$E(X) = \int_0^1 \int_0^2 xf(x,y)dxdy = \int_0^1 \int_0^2 x\frac{x(1+3y^2)}{4}dxdy = \frac{4}{3}$$

$$E(Y) = \int_0^1 \int_0^2 yf(x,y)dxdy = \int_0^1 \int_0^2 y\frac{x(1+3y^2)}{4}dxdy = \frac{5}{8}$$

$$(E(XY) =)\frac{5}{6} = \frac{4}{3} * \frac{5}{8}(= E(X) * E(Y))$$

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If *a* and *b* are constants, then

$$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If a and b are constants, then

  $$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If *a* and *b* are constants, then

  $$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$
  - The variance is unchanged if a constant is added to or subtracted from a random variable.

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If a and b are constants, then

  $$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$
  - The variance is unchanged if a constant is added to or subtracted from a random variable.
  - The addition or subtraction of a constant simply shifts the values of X to the right/left but does not change their variability.

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If a and b are constants, then

  $$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$
  - The variance is unchanged if a constant is added to or subtracted from a random variable.
  - The addition or subtraction of a constant simply shifts the values of X to the right/left but does not change their variability.
- Corollary 4.7: $\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$

Means and Variance of Linear Combinations of Random Variables VI

- **Theorem 4.9**:

  If a and b are constants, then

  $$\sigma^2_{aX+b} = a^2\sigma^2_X = a^2\sigma^2$$

- Corollary 4.6: $\sigma^2_{X+b} = \sigma^2_X = \sigma^2$
  - The variance is unchanged if a constant is added to or subtracted from a random variable.
  - The addition or subtraction of a constant simply shifts the values of X to the right/left but does not change their variability.
- Corollary 4.7: $\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2$
  - The variance is multiplied or divided by the square of the constant.

Means and Variance of Linear Combinations of Random Variables VII

- **Theorem 4.10**:

  If X and Y are random variables with joint probability distribution $f(x, y)$, then

  $$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

Means and Variance of Linear Combinations of Random Variables VII

- **Theorem 4.10**:
  If X and Y are random variables with joint probability distribution $f(x, y)$, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

- Corollary 4.8: If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

Means and Variance of Linear Combinations of Random Variables VII

- **Theorem 4.10**:
  If X and Y are random variables with joint probability distribution $f(x, y)$, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

- Corollary 4.8: If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- Corollary 4.9: If X and Y are independent random variables, then

$$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

Means and Variance of Linear Combinations of Random Variables VII

- **Theorem 4.10**:

  If X and Y are random variables with joint probability distribution $f(x, y)$, then

  $$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}$$

- Corollary 4.8: If X and Y are independent random variables, then

  $$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- Corollary 4.9: If X and Y are independent random variables, then

  $$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y$$

- Corollary 4.10: If $X_1, X_2, \ldots X_n$ are independent random variables, then

  $$\sigma^2_{a_1X_1+a_2X_2+\ldots a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \ldots + a_n^2\sigma^2_{X_n}$$

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \text{ (by Theorem 4.9)}$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \text{ (by Theorem 4.10)}$$
$$= 130$$

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \ (by\ Theorem\ 4.9)$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \ (by\ Theorem\ 4.10)$$
$$= 130$$

- **Example 4.21**: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product.

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \ (by\ Theorem\ 4.9)$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \ (by\ Theorem\ 4.10)$$
$$= 130$$

- **Example 4.21**: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product.
- Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 3$

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \ (by\ Theorem\ 4.9)$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \ (by\ Theorem\ 4.10)$$
$$= 130$$

- **Example 4.21**: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product.
- Suppose that X and Y are <u>independent random variables</u> with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 3$
- Find the variance of the random variable $Z = 3X - 2Y + 5$

Means and Variance of Linear Combinations of Random Variables VIII

- **Example 4.20**: X and Y are random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 2$, and covariance $\sigma_{XY} = -2$,
- Find the variance of the random variable $Z = 3X - 4Y + 8$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \ (by\ Theorem\ 4.9)$$
$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \ (by\ Theorem\ 4.10)$$
$$= 130$$

- **Example 4.21**: Let X and Y denote the amount of two different types of impurities in a batch of a certain chemical product.
- Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$, $\sigma_Y^2 = 3$
- Find the variance of the random variable $Z = 3X - 2Y + 5$
- Solution:

$$\sigma_Z^2 = \sigma_{3X-2Y+5}^2 = \sigma_{3X-2Y}^2 \ (by\ Theorem\ 4.9)$$
$$= 9\sigma_X^2 + 4\sigma_Y^2 \ (by\ Corollary\ 4.9)$$
$$= 30$$

Chebyshev's Theorem I

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean

Chebyshev's Theorem I

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean
- A large variance indicates a greater variability, so the area of distribution should be spread out more.

Chebyshev's Theorem I

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean
- A large variance indicates a greater variability, so the area of distribution should be spread out more.

Chebyshev's Theorem I

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean
- A large variance indicates a greater variability, so the area of distribution should be spread out more.

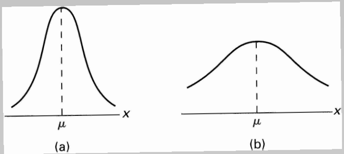

Figure: Variability of continuous observations about the mean.

Chebyshev's Theorem I

- If a random variable has a small variance or standard deviation, we would expect most of the values to be grouped around the mean
- A large variance indicates a greater variability, so the area of distribution should be spread out more.

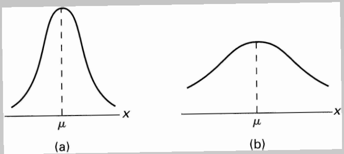

Figure: Variability of continuous observations about the mean.

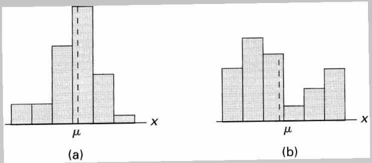

Figure: Variability of discrete observations about the: mean.

Chebyshev's Theorem II

- **Theorem 4.11**:

  (**Chebyshev's theorem**) The probability that any random variable X will assume a value within k standard deviation of the mean is at least $1 - 1/k^2$. That is

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

Chebyshev's Theorem II

- **Theorem 4.11**:

  (**Chebyshev's theorem**) The probability that any random variable X will assume a value within k standard deviation of the mean is at least $1 - 1/k^2$. That is

  $$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

- **Example 4.22**: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find

Chebyshev's Theorem II

- **Theorem 4.11**:

  (**Chebyshev's theorem**) The probability that any random variable X will assume a value within k standard deviation of the mean is at least $1 - 1/k^2$. That is

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

- **Example 4.22**: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find
- $P(-4 < X < 20)$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(-4 < X < 20) = P(8 - 4 * 3 < X < 8 + 4 * 3) \geq 1 - \frac{1}{4^2} = \frac{15}{16}$$

Chebyshev's Theorem II

- **Theorem 4.11**:

  (**Chebyshev's theorem**) The probability that any random variable X will assume a value within k standard deviation of the mean is at least $1 - 1/k^2$. That is

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

- **Example 4.22**: A random variable X has a mean $\mu = 8$, a variance $\sigma^2 = 9$, and an unknown probability distribution. Find
- $P(-4 < X < 20)$

$$P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - \frac{1}{k^2}$$

$$P(-4 < X < 20) = P(8 - 4*3 < X < 8 + 4*3) \geq 1 - \frac{1}{4^2} = \frac{15}{16}$$

- $P(|X - 8| \geq 6)$

$$P(|X - 8| \geq 6) = 1 - P(|X - 8| < 6) = 1 - P(-6 < X - 8 < 6)$$

$$= 1 - P(8 - 6 < X < 6 + 8) = 1 - P(8 - 2*3 < X < 8 + 2*3) \leq \frac{1}{2^2} = \frac{1}{4}$$

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.
- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.
- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.
- The use of Chebyshev's theorem;

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.
- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.
- The use of Chebyshev's theorem;
  - holds for any distribution of observations

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.
- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.
- The use of Chebyshev's theorem;
  - holds for any distribution of observations
  - gives a lower bound only

Chebyshev's Theorem III

- The Chebyshev inequality is a useful tool as well as a relation that connects the variance of a distribution with the intuitive notation of dispersion in a distribution.
- For any population or sample, this provides that the minimum probability of the data within $k\sigma$ from the mean μ is $1 - \frac{1}{k^2}$.
- The use of Chebyshev's theorem;
  - holds for any distribution of observations
  - gives a lower bound only
  - is suitable to situations where the form of the distribution is unknown (a distribution-free result)