Lecture 8
Some Discrete Probability Distributions II
Lecture Information

Ceng272 *Statistical Computations* at April 12, 2010

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Contents

Introduction and Motivation

- **Negative binomial distribution (Geometric distribution)**: the number of trial on which the first success occurs.

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Introduction and Motivation

- **Negative binomial distribution (Geometric distribution)**: the number of trial on which the first success occurs.
- **Poisson distribution**: the number of outcomes occurring during a given time interval or in a specified region.

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- **Negative binomial random variable**: the number X of trials to produce k success in a negative binomial experiment.
- **Negative binomial distribution**: If repeated independent trials can result in a success with probability p and a failure with probability $q = 1 - p$, then the probability distribution of the random variable X, the number of the trial on which the k^{th} success occurs, is

$$b^*(x; k, p) = \binom{x-1}{k-1} p^k q^{x-k}, \; x = k, k+1, k+2, \ldots$$

Negative Binomial and Geometric Distributions II

- **Example 5.17**: Suppose that team *A* has probability 0.55 of winning over the team *B* and both teams *A* and *B* face each other in an NBA 4-out-of-7 championship series.

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$$b^*(x;k,p) = \qquad b^*(6;4,0.55) = \left(\begin{array}{c}5\\3\end{array}\right) * 0.55^4 * (1*0.55)^{6-4} = 0.1853$$

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$$b^*(4;4,0.55) + b^*(5;4,0.55) + b^*(6;4,0.55) + b^*(7;4,0.55)$$
$$= 0.0915 + 0.1647 + 0.1853 + 0.1668 = 0.6083$$

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$$b^*(3;3,0.55) + b^*(4;3,0.55) + b^*(5;3,0.55)$$
$$= 0.1664 + 0.2246 + 0.2021 = 0.5931$$

Negative Binomial and Geometric Distributions III

- **Why name negative binomial?** The binomial coefficient is defined even when *n* is negative (or is not an integer).

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

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- Each term in the expansion of $p^k(1-q)^{-k}$ corresponds to the value of $b^*(x;k,p)$ for $x = k, k+1, k+2, \dots$

$$1 = p^k * p^{-k} = p^k * (1-q)^{-k} = p^k * \sum_{x=0}^{\infty} \binom{-k}{x} (-q)^x$$

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  $$E(Y) = 1/0.35 = 2.85.$$
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$$P(X = x) = g(5; 0.05) = 0.05 * 0.95^4 = 0.041$$

Negative Binomial and Geometric Distributions VIII

- **Theorem 5.4**:

  The mean and variance of a random variable following the geometric distribution are

$$\mu = \frac{1}{p}, \quad \sigma^2 = \frac{1-p}{p^2}$$

Negative Binomial and Geometric Distributions VIII

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- A high probability of requiring a large of number of attempts is not beneficial to the scientists or engineers.

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- The examples of the random variables that are having Poisson probability distribution are usually rare events such as # of car accident, # of flood or hurricane occurrences.
- So the Poisson distribution provides the fundamental idea of the Law of Small Numbers (LSN).

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  1. The number of outcomes in one time interval or specified region is independent of the number that occurs in any other disjoint time interval or region of space. In this way we say that the Poisson process has no memory.

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  2. The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.
  3. The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

Poisson Distribution and the Poisson Process III

- **Poisson Distribution**: The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is (mean number $\mu = \lambda t$)

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$$

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- **Example 5.20**: During a laboratory experiment the average number of radioactive particles passing through a counter in 1 millisecond is 4.
- What is the probability that 6 particles enter the counter in a given millisecond? (see Table A.2)

$$p(6; 4) = \frac{e^{-4}(4)^6}{6!} = \sum_{x=0}^{6} p(x; 4) - \sum_{x=0}^{5} p(x; 4)$$

$$= 0.8893 - 0.7851 = 0.1042$$

Poisson Distribution and the Poisson Process IV

TABLE A.2 Poisson Probability Sums $\sum_{x=0}^{r} p(x;\mu)$

	μ								
r	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**
0	0.9048	0.8187	0.7408	0.6730	0.6065	0.5488	0.4966	0.4493	0.4066
1	0.9953	0.9825	0.9631	0.9384	0.9098	0.8781	0.8442	0.8088	0.7725
2	0.9998	0.9989	0.9964	0.9921	0.9856	0.9769	0.9659	0.9526	0.9371
3	1.0000	0.9999	0.9997	0.9992	0.9982	0.9966	0.9942	0.9909	0.9865
4		1.0000	1.0000	0.9999	0.9998	0.9996	0.9992	0.9986	0.9977
5				1.0000	1.0000	1.0000	0.9999	0.9998	0.9997
6							1.0000	1.0000	1.0000

	μ								
r	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**	**4.5**	**5.0**
0	0.3679	0.2231	0.1353	0.0821	0.0498	0.0302	0.0183	0.0111	0.0067
1	0.7358	0.5578	0.4060	0.2873	0.1991	0.1359	0.0916	0.0611	0.0404
2	0.9197	0.8088	0.6767	0.5438	0.4232	0.3208	0.2381	0.1736	0.1247
3	0.9810	0.9344	0.8571	0.7576	0.6472	0.5366	0.4335	0.3423	0.2650
4	0.9963	0.9814	0.9473	0.8912	0.8153	0.7254	0.6288	0.5321	0.4405
5	0.9994	0.9955	0.9834	0.9580	0.9161	0.8576	0.7851	0.7029	0.6160
6	0.9999	0.9991	0.9955	0.9858	0.9665	0.9347	0.8893	0.8311	0.7622
7	1.0000	0.9998	0.9989	0.9958	0.9881	0.9733	0.9489	0.9134	0.8666
8		1.0000	0.9998	0.9989	0.9962	0.9901	0.9786	0.9597	0.9319
9			1.0000	0.9997	0.9989	0.9967	0.9919	0.9829	0.9682
10				0.9999	0.9997	0.9990	0.9972	0.9933	0.9863
11				1.0000	0.9999	0.9997	0.9991	0.9976	0.9945
12					1.0000	0.9999	0.9997	0.9992	0.9980
13						1.0000	0.9999	0.9997	0.9993
14							1.0000	0.9999	0.9998
15								1.0000	0.9999
16									1.0000

Figure: Poisson Probability Sums $P(x;\mu) = \sum_{x=0}^{r} p(x;\mu)$.

Poisson Distribution and the Poisson Process V

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Poisson Distribution and the Poisson Process V

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- Using Chebyshev's theorem, we conclude that at least 3/4 of the time the number of radioactive particles entering the counter will be anywhere from 0 to 8 during a given millisecond.

Poisson Distribution and the Poisson Process VI

- **The Poisson Distribution As a Limiting Form of the Binomial**: **Theorem 5.6**:

Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty, p \to 0$, and $\mu = np$ remains constant,

$$b(x; n, p) \xrightarrow{n\to\infty} p(x; \mu)$$

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  (Proof is in Appendix A.27)

- If $p \to 1$, we can change p to a value close to 0 by interchanging what we have defined to be a success and a failure.

Poisson Distribution and the Poisson Process VII

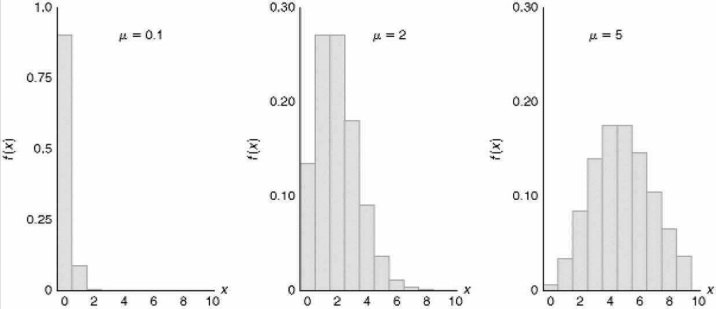

Figure: Poisson density functions for different means.

The shape becomes more symmetric as the mean grows large.

Poisson Distribution and the Poisson Process VIII

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Poisson Distribution and the Poisson Process VIII

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Poisson Distribution and the Poisson Process VIII

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$$P(X = 1) = e^{-2}2^{1} = 0.271$$

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$$P(X \leq 3) = \sum_{x=0}^{3} \frac{e^{-2}2^{x}}{x!} = 0.857$$

Poisson Distribution and the Poisson Process IX

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- What is the probability that a random sample of 8000 will yield fewer than 7 items processing bubbles?

Poisson Distribution and the Poisson Process IX

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- It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles.
- What is the probability that a random sample of 8000 will yield fewer than 7 items processing bubbles?
- Solution:

$$n = 8000, p = 0.001 \Rightarrow \mu = np = 8$$

$$P(X \leq 7) = \sum_{x=0}^{6} b(x; 8000, 0.001) \approx \sum_{x=0}^{6} p(x; 8) = 0.3134$$