Table 1: CENG375 Numerical Computations - Formulae Sheet

$y_{calc} = f(x_{calc})$
$E_{fwd} = y_{calc} - y_{exact}$
$E_{backwd} = x_{calc} - x,\ y_{calc} = f(x_{calc})$

$P_n(x) = f(x) = f(a) + \frac{f^{'}(a)}{1!}(x-a) + \frac{f^{''}(a)}{2!}(x-a)^2 + \frac{f^{'''}(a)}{3!}(x-a)^3 + \ldots + \frac{f^n(a)}{n!}(x-a)^n + \ldots$

$Error\ of\ TS = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(\xi), where\ \xi\ in\ [a, x]$

Algorithm: Bisection Method

To determine a root of $f(x) = 0$ that is accurate within a specified tolerance value, given values x_1 and x_2, such that $f(x_1) * f(x_2) < 0$,
Repeat
Set $x3 = (x_1 + x_2)/2$
If $f(x_3) * f(x_1) < 0$ Then
Set $x_2 = x_3$
Else Set $x_l = x_3$ End If
Until $(|x_1 - x_2|) < 2 * tolerance\ value$

Algorithm: Secant Method

To determine a root of $f(x) = 0$, given two values, x_0 and x_1, that are near the root,
If$|f(x_0)| < |f(x_1)|$ Then
Swap x_0 with x_1
Repeat
Set $x_2 = x_1 - f(x_1) * \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$
Set $x_0 = x_1$, Set $x_1 = x_2$
Until $|f(x_2)| < tolerance\ value$

Algorithm: False Position Method

To determine a root of $f(x) = 0$, given two values of x_0 and x_1 that bracket a root: that is, $f(x_0)$ and $f(x_1)$ are of opposite sign,
Repeat
Set $x_2 = x_1 - f(x_1) * \frac{(x_0 - x_1)}{f(x_0) - f(x_1)}$
If $f(x_2)$ is of opposite sign to $f(x_0)$ Then
Set $x_1 = x_2$,
Else
Set $x_0 = x_2$
End If
Until $|f(x_2)| < tolerance\ value$.

Algorithm: Newton Method

To determine a root of $f(x) = 0$, given x_0 reasonably close to the root,
Compute $f(x_0), f^{'}(x_0)$
If $(f(x_0) \neq 0)$ And $(f^{'}(x_0) \neq 0)$ Then
Repeat
Set $x_1 = x_0$
Set $x_0 = x_0 - \frac{f(x_0)}{f^{'}(x_0)}$
Until $(|x_1 - x_0| < tolerance\ value1)$
Or If $|f(x_0)| < tolerance\ value2)$
End If.

$error\ after\ n\ iterations < \frac{(b-a)}{2^n}$

$x_{n+1} = x_n - f(x_n)\frac{(x_{n-1} - x_n)}{f(x_{n-1}) - f(x_n)}, \quad n = 0, 1, 2, \ldots$

$x_{n+1} = x_n - \frac{f(x_n)}{f^{'}(x_n)}, \quad n = 0, 1, 2, \ldots$

Table 2: Formulae Sheet Cont.

Algorithm: Muller Method : Given the points x_2, x_0, x_1 in increasing value, Evaluate the corresponding function values: f_2, f_0, f_1. Repeat (Evaluate the coefficients of the parabola, $a\nu^2 + b\nu + c$, determined by the three points. $(x_2, f_2), (x_0, f_0), (x_l, f_1)$.) Set $h_l = x_1 - x_0; h_2 = x_0 - x_2; \gamma = h_2/h_1$. Set $c = f_0$ Set $a = \frac{\gamma f_1 - f_0(1+\gamma) + f_2}{\gamma h_1^2(1+\gamma)}$ Set $b = \frac{f_1 - f_0 - ah_1^2}{h_1}$ (Next, compute the roots of the polynomial.) Set $root = x_0 - \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$ Choose root, x_r, closest to x_0 by making the denominator as large as possible; i.e. if $b > 0$, choose plus; otherwise, choose minus. If $x_r > x_0$, Then rearrange to:x_0, x_1, and the root Else rearrange to: x_0, x_2, and the root End If. (In either case, reset subscripts so that x_0, is in the middle.) Until $\|f(x_r)\| < Ftol$	$\nu = x - x_0$, $h_1 = x_1 - x_0$, $h_2 = x_0 - x_2$ $\gamma = h_2/h_1$, $a(0)^2 + b(0) + c = f_0$ $ah_1^2 + bh_1 + c = f_1$ $ah_2^2 - bh_2 + c = f_2$ $c = f_0$, $a = \frac{\gamma f_1 - f_0(1+\gamma) + f_2}{\gamma h_1^2(1+\gamma)}$, $b = \frac{f_1 - f_0 - ah_1^2}{h_1}$, $\nu_{1,2} = \frac{2c}{b \pm \sqrt{b^2 - 4ac}}$, $root = x_0 - \nu$
Algorithm: Fixed Point Method To determine a root of $f(x) = 0$, given a value x_1 reasonably close to the root Rearrange the equation to an equivalent form $x = g(x)$ Repeat Set $x_2 = x_l$ Set $x_l = g(x_1)$ Until $\|x_1 - x_2\| < tolerance\ value$	$x_{n+1} = g(x_n); \quad n = 0, 1, 2, 3, \ldots$
$f_x(x_i, y_i)\Delta x_i + f_y(x_i, y_i)\Delta y_i = -f(x_i, y_i)$ $g_x(x_i, y_i)\Delta x_i + g_y(x_i, y_i)\Delta y_i = -g(x_i, y_i)$	$x_{i+1} = x_i + \Delta x_i$ $y_{i+1} = y_i + \Delta y_i$

$$Ax = b \Longrightarrow A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Table 3: Formulae Sheet Cont.

$$A = L * U \Longrightarrow A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ (X) & 1 & 0 \\ (X) & (X) & 1 \end{bmatrix}}_{L} * \underbrace{\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{bmatrix}}_{U} \quad , \quad Ax = b \; LUx = b \; Ly = b$$

$$A^{-1}Ax = A^{-1}b \mapsto x = A^{-1}b$$

$$||x||_p = \left(\textstyle\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \; , \quad \begin{aligned} ||x||_1 &= \textstyle\sum_{i=1}^{n} |x_i| = sum\ of\ magnitudes \\ ||x||_2 &= \left(\textstyle\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = Euclidean\ norm \\ ||x||_\infty &= max_{1\le i\le n}|x_i| = maximum - magnitude\ norm \end{aligned}$$

$$\begin{aligned} ||A||_1 &= max_{1\le j\le n} \textstyle\sum_{i=1}^{n} |a_{ij}| = maximum\ column\ sum \\ ||A||_\infty &= max_{1\le i\le n} \textstyle\sum_{j=1}^{n} |a_{ij}| = maximum\ row\ sum \end{aligned} \quad , \; ||A||_f = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

$$|a_{ii}| > \textstyle\sum_{j=1, j\ne i}^{n} |a_{ij}|, i = 1, 2, \ldots, n, \quad diagonally\ \ dominant$$

$$x_i = \frac{b_i}{a_{ii}} - \sum_{j=1, j\ne i}^{n} \frac{a_{ij}}{a_{ii}} x_j, i = 1, 2, \ldots, n$$

$$F(x) = a_1\Phi_1(x) + a_2\Phi_2(x) + \ldots + a_n\Phi_n(x); \quad F(x) = a_1 + a_2x + a_3x^2 + \ldots + a_nx^{n-1}$$

$$P_{n-1}(x) = y_1L_1(x) + y_2L_2(x) + \ldots + y_nL_n(x) = \textstyle\sum_{j=1}^{n} y_jL_j(x)$$

$$L_j(x) = \prod_{k=1, k\ne j}^{n} \frac{x - x_k}{x_j - x_k}$$

$$f(x) = \frac{(x - x_2)}{(x_1 - x_2)} f_1 + \frac{(x - x_1)}{(x_2 - x_1)} f_2$$

$$p_{i,j} = \frac{(x - x_i) * P_{i+1, j-1} + (x_{i+j} - x) * P_{i, j-1}}{x_{i+j} - x_i}$$

$$P_n(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + (x - x_0)(x - x_1)\ldots(x - x_{n-1})a_n$$

$$f[x_s] = f_s; \;\; f[x_s, x_t] = \frac{f_t - f_s}{x_t - x_s}; \;\; f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}; \;\; f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$f[x_0, x_1, \ldots, x_n] = \frac{f[x_1, x_2, \ldots, f_n] - f[x_0, x_1, \ldots, f_{n-1}]}{x_n - x_0}$$

$$P_n(x_i) = f_i\ for\ i = 0, 1, 2, \ldots, n.$$

$$P_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] +$$
$$(x - x_0)(x - x_1)(x - x_2)f[x_0, \ldots, x_3] + (x - x_0)(x - x_1)\ldots(x - x_{n-1})f[x_0, \ldots, x_n]$$

$$P_3(x) = f[x_0] + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2]$$
$$+(x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3]$$

$$g_i(x_i) = y_i, \; i = 0, 1, \ldots, n-1, \; g_{n-1}(x_n) = y_n; \;\; g_i(x_{i+1}) = g_{i+1}(x_{i+1}), \; i = 0, 1, \ldots, n-2;$$
$$g_i^{'}(x_{i+1}) = g_{i+1}^{'}(x_{i+1}), \; i = 0, 1, \ldots, n-2; \;\; g_i^{''}(x_{i+1}) = g_{i+1}^{''}(x_{i+1}), \; i = 0, 1, \ldots, n-2$$

$$e_i = Y_i - y_i; \quad S \;\; = e_1^2 + e_2^2 + \ldots + e_n^2 = \textstyle\sum_{i=1}^{N} e_i^2; \quad \partial S/\partial a \qquad \& \qquad \partial S/\partial b$$

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$$

$$\overbrace{\begin{bmatrix} N & \sum x_i & \sum x_i^2 & \sum x_i^3 & \ldots & \sum x_i^n \\ \sum x_i & \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \ldots & \sum x_i^{n+1} \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 & \sum x_i^5 & \ldots & \sum x_i^{n+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum x_i^n & \sum x_i^{n+1} & \sum x_i^{n+2} & \sum x_i^{n+3} & \ldots & \sum x_i^{2n} \end{bmatrix}}^{B} \overbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}^{a} = \begin{bmatrix} \sum Y_i \\ \sum x_iY_i \\ \sum x_i^2Y_i \\ \vdots \\ \sum x_i^nY_i \end{bmatrix}$$

Table 4: Formulae Sheet Cont.

$\overbrace{\begin{bmatrix}1&1&1&1&1\\x_1&x_2&x_3&\dots&x_N\\x_1^2&x_2^2&x_3^2&\dots&x_N^2\\\vdots&\vdots&\vdots&\vdots&\vdots\\x_1^n&x_2^n&x_3^n&\dots&x_N^n\end{bmatrix}}^{A}\overbrace{\begin{bmatrix}y_1\\y_2\\y_3\\\vdots\\y_n\end{bmatrix}}^{y}=\begin{bmatrix}\sum Y_i\\\sum x_iY_i\\\sum x_i^2Y_i\\\vdots\\\sum x_i^nY_i\end{bmatrix};\ AA^Ta=Ba=Ay$
$\int_a^b P_n^*(x)P_m(x)dx=0\ when\ n\neq m$ $\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx=\left\{\begin{array}{ll}0, & n\neq m\\ \pi, & n=m=0\\ \pi/2, & n=m\neq 0\end{array}\right\}$ $T_{n+1}(x)=2xT_n(x)-T_{n-1}(x);\quad T_0(x)=1\&\ T_1(x)=x$ ES=TS-CS; $\frac{1}{2^{n-1}}T_n(x)$
$f(x)=f(x+P)=f(x+2P)=\ldots=f(x-P)=f(x-2P)=\ldots$ $f(x)\approx\frac{A_0}{2}+\sum_{n=1}^{\infty}[A_ncos(nx)+B_nsin(nx)]$ $A_n=\frac{1}{P/2}\int_{-P/2}^{P/2}f(x)cos\left(\frac{\pi}{P/2}nx\right)dx,\ n=0,1,2,\ldots$ $B_n=\frac{1}{P/2}\int_{-P/2}^{P/2}f(x)sin\left(\frac{\pi}{P/2}nx\right)dx,\ n=1,2,3,\ldots$
$f(x)\ is\ even\ if\ f(-x)=f(x);\ \ f(x)\ is\ odd\ if\ f(-x)=-f(x)$ $if\ f(x)\ is\ even,\ \ \int_{-L}^{L}f(x)dx=2\int_0^L f(x)dx$ $if\ f(x)\ is\ odd,\ \ \int_{-L}^{L}f(x)dx=0$ $A_n=\frac{2}{L}\int_0^L f(x)cos\left(\frac{n\pi x}{L}\right)dx,\ n=0,1,2,\ldots$ $B_n=\frac{2}{L}\int_0^L f(x)sin\left(\frac{n\pi x}{L}\right)dx,\ n=1,2,3,\ldots$
$\frac{df}{dx}\big\rvert_{x=a}=lim_{\Delta x\to 0}\frac{f(a+\Delta x)-f(a)}{\Delta x}$ $\frac{df}{dx}\big\rvert_{x=a}=\frac{f(a+\Delta x)-f(a)}{\Delta x};\ \ \frac{df}{dx}\big\rvert_{x=a}=\frac{f(a)-f(a-\Delta x)}{\Delta x};\ \ f'(x)=\frac{f(x+h)-f(x-h)}{2h}-f'''(\xi)*\frac{h^2}{6}$
$F'(x)=f(x);\ \int_a^b f(x)dx=F(b)-F(a)$ $\int_{x_i}^{x_{i+1}}f(x)dx\approx\frac{f_i+f_{i+1}}{2}(x_{i+1}-x_i);\ \ Error=-(1/12)h^3f''(\xi)=O(h^3)$ $\int_a^b\approx\sum_{i=0}^{n-1}\frac{h}{2}(f_i+f_{i+1})=\frac{h}{2}(f_0+2f_1+2f_2+\ldots+2f_{n-1}+f_n)$ $Global\ error=(-1/12)h^3nf''(\xi)=O(h^3);\ or$ $Global\ error=\frac{-(b-a)}{12}h^2f''(\xi)=O(h^2)$ where ξ in [a, b]