1 Solving Sets of Equations

- Solving sets of linear equations is the most frequently used numerical procedure when real-world situations are modelled.
- The methods for solving ordinary and partial-differential equations depend on them.

1. **Matrices and Vectors**. Reviews concepts of matrices and vectors in preparation for their use.
2. **Elimination Methods.** Describes two classical methods that change a system of equations to forms that allow getting the solution by back-substitution and shows how the errors of the solution can be minimized.

3 **The Inverse of a Matrix and Matrix Pathology.** Shows how an important derivative of a matrix, its inverse, can be computed. It shows when a matrix cannot be inverted and tells of situations where no unique solution exists to a system of equations.

4 **Ill-Conditioned Systems.** Explores systems for which getting the solution with accuracy is very difficult. A number, the condition number, is a measure of such difficulty; a property of a matrix, called its norm, is used to compute its condition number. A way to improve an inaccurate solution is described.

5 **Iterative Methods.** It is described how a linear system can be solved in an entirely different way, by beginning with an initial estimate of the solution and performing computations that eventually arrive at the correct solution. An iterative method is particularly important in solving systems that have few nonzero coefficients.

1.1 Matrices and Vectors

- When a system of equations has more than two or three equations, it is difficult to discuss them without using *matrices* and *vectors*.
- A *matrix* is a rectangular array of numbers in which not only the value of the number is important but also its position in the array.

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} = [a_{ij}] \,, \ i = 1, 2, \dots, n, \ j = 1, 2, \dots, m$$

- In general, $AB \neq BA$ (B is another matrix), so the order of factors must be preserved in matrix multiplication.
- A matrix
  - with only one column, $n \times 1$ in size, is termed a **column vector**,
  - with only one row, $1 \times m$ in size, is called a **row vector**.
- When the term **vector** is used, it nearly always means a **column** vector.
- An $m \times n$ matrix times as $n \times 1$ vector gives an $m \times 1$ product ($m \times n n \times 1$) .
- The general relation for $Ax = b$ is

$$b_i = \sum_{k=1}^{No.of.cols.} a_{ik} x_k, \qquad i = 1, 2, \ldots, No.of.rows$$

  where A is a matrix, x and b are vectors (column vectors).
- Set of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

- Much more simply in matrix notation, as $Ax = b$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

- Example,

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 0 \\ -1 & 3 & 2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ -7 \\ 2 \end{bmatrix}$$

- is the same as the set of equations

$$\begin{aligned} &3x_1 + 2x_2 + 4x_3 = 14 \\ &x_1 - 2x_2 = -7 \\ &-x_1 + 3x_2 + 2x_3 = 2 \end{aligned}$$

1.1.1 Some Special Matrices and Their Properties

- Square matrices are particularly important when a system of equations is to be solved. (special properties).
- The elements on the *main diagonal* are those from the upper-left corner to the lower-right corner. (diagonal elements, diagonal)
- If the nonzero elements of a matrix diagonal all are equal to one, the matrix is called the *identity matrix of order n* where n equals the number of row and columns.
- i.e., The order-4 identity matrix is

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- An important property of an identity matrix, I, is that for any $n \times n$ (square) matrix, A,

$$I * A = A * I = A$$

- If two rows of an identity matrix are interchanged, it is called a *transposition matrix*. (We also get a transposition matrix by interchanging two columns).
- If transposition matrix P_1, is multiplied with a square matrix of the same size, A, the product $P_1 * A$ will be the A matrix but with the same two rows interchanged. i.e.:

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 4 & 2 & 8 & 11 \\ 0 & 7 & 1 & 9 \\ 3 & 2 & 6 & 8 \end{bmatrix},$$

$$P_1 * A = \begin{bmatrix} 9 & 6 & 2 & 13 \\ 3 & 2 & 6 & 8 \\ 0 & 7 & 1 & 9 \\ 4 & 2 & 8 & 11 \end{bmatrix},$$

- However, if the two matrices are multiplied in reverse order, $A * P_1$, the result will be matrix A but with the columns of A interchanged.
- A permutation matrix is obtained by multiplying several transposition matrices.
- **Symmetric matrix**

  A square matrix is called a **symmetric matrix** when the pairs of elements in similar positions across the diagonal are equal.

$$\begin{bmatrix} 1 & x & y \\ x & 2 & z \\ y & z & 3 \end{bmatrix}$$

- The transpose of a matrix is the matrix obtained by writing the rows as columns or by writing the columns as rows.
- A matrix does not have to be square to have a transpose.
- The symbol for the transpose of matrix A is A^T.

$$A = \begin{bmatrix} 3 & -1 & 4 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix},$$

$$A^T = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 1 \\ 4 & -3 & 2 \end{bmatrix}$$

  - if A is symmetric $\Longrightarrow A = A^T$,
  - For any matrix, the transpose of the transpose, $(A^T)^T$, is just A itself,
  - It is also true that $(A * B)^T = B^T * A^T$.

- When a matrix is square, a quantity called its **trace** is defined.
- The trace of a square matrix is the **sum of the elements on its main diagonal**.
- The trace remains the same if a square matrix is transposed $\Longrightarrow tr(A) = tr(A^T)$

- If all the elements above the diagonal are zero, a matrix is called *lower-triangular* (L);
- It is called *upper-triangular* (U) when all the elements below the diagonal are zero.

$$L = \begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix},$$

$$U = \begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}$$

- Tridiagonal matrices;

$$\begin{bmatrix} x & x & 0 & 0 & 0 \\ x & x & x & 0 & 0 \\ 0 & x & x & x & 0 \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

  have nonzero elements only on the diagonal and in the positions adjacent to the diagonal. Only the nonzero values need to be recorded; $n \times n \Longrightarrow n \times 3$.

- In some important applied problems, only a few of the elements are nonzero.
- Such a matrix is termed a **sparse matrix** and procedures that take advantage of this sparseness are of value.
- Division of matrices is not defined, but we will discuss the inverse of a matrix.
- The determinant of a square matrix is a number.
  - The method of calculating determinants is a lot of work if the matrix is of large size.
  - Methods that triangularize a matrix, as described in Section 1.2, are much better ways to get the determinant.

- If a matrix, B, is triangular (either upper or lower), its determinant is just the product of the diagonal elements:

$$det(B) = \Pi B_{ii}, \qquad i = 1, \ldots, n$$

$$det \begin{bmatrix} 4 & 0 & 0 \\ 6 & -2 & 0 \\ 1 & -3 & 5 \end{bmatrix} = -40$$

- Determinants can be used to obtain the **characteristic polynomial** and the **eigenvalues** of a matrix, which are the roots of that polynomial.

- Eigenvalue is a German word, the corresponding English term is characteristic value, but it is less frequently used.

- For matrix A,

$$A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$$

$$P_A(\lambda) = |A - \lambda I| = det \begin{bmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{bmatrix} = \lambda^2 - 6\lambda - 7$$

- The characteristic polynomial is always of degree n if A is $n \times n$.

- If we set the characteristic polynomial to zero and solve for the roots, we get the eigenvalues of A.

- For this example, these are $\lambda_1 = 7$, $\lambda_2 = -1$, or, in more symbolic mathematical notation:

$$\Lambda(A) = \{7, -1\}$$

- We also mention the notion of an **eigenvector** corresponding to an eigenvalue.

$$Aw = \lambda w \Longrightarrow$$

$$(A - \lambda I)w = 0$$

$$w_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, w_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

In the current example, the eigenvectors are given as above.

- Observe that the trace of A is equal to the sum of the eigenvalues: $tr(A) = 1 + 5 = \lambda_1 + \lambda_2 = 7 + (-1) = 6$.
- This is true for any matrix: **The sum of its eigenvalues equals its trace**.
- If a matrix is triangular, **its eigenvalues are equal to the** *diagonal elements*.
- This follows from the fact that
  - its determinant is just the product of the diagonal elements and
  - its characteristic polynomial is the product of the terms $(a_{ii} - \lambda)$ with i going from 1 to n, the number of rows of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix},$$

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda)(6-\lambda)$$

whose roots are clearly 1, 4, and 6.

- It does not matter if the matrix is upper- or lower-triangular.

1.1.2 Using Computer Algebra Systems

MATLAB can do matrix operations . We first define two matrices and a vector, A, B, and ν:

1.2 Elimination Methods

- To solve a set of linear equations.
- The term linear equation means an equation in several variables where all of the variables occur to the first power.
- Suppose we have a system of equations that is of a special form, an *upper-triangular* system, such as

$$\begin{aligned} 5x_1 + 3x_2 - 2x_3 &= -3 \\ 6x_2 + x_3 &= -1 \\ 2x_3 &= 10 \end{aligned}$$

we have the solution $x_1 = 2,\ x_2 = -1,\ x_3 = 5$

```
>> A = [ 4 1 -2 ; 5 1 3 ; 4 0 -1]
>> B = [ 3 3 1 ; -2 1 5 ; 2 2 0]
>> v = [ -2 3 1]
>> vt = v'
>> det(A)
>> trace (A)
We can get the characteristic polynomial;
>> poly(A)
ans =
1.000   -4.0000   2.0000   -21.0000
where the coefficients are given. This represents
x^3 - 4x^2 + 2x -21
```

- Change the matrix of coefficients $\Longrightarrow$ upper triangular.

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -3x_1 - x_2 + 4x_3 &= 8 \\ x_1 - x_2 + 3x_3 &= 13 \end{aligned}$$

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -2x_2 + 11x_3 &= 37 \end{aligned}$$

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -72x_3 &= -216 \end{aligned}$$

Now we have a triangular system and the solution is readily obtained;

1. obviously $x_3 = 3$ from the third equation,
2. and **back-substitution** into the second equation gives $x_2 = -2$.
3. We continue with back-substitution by substituting both x_2, and x_3 into the first equation to get $x_1 = 2$.

- Notice to the values in the previous example.
- They are getting bigger!
- The essence of any elimination method is to reduce the coefficient matrix to a triangular matrix and then use back-substitution to get the solution.

- We now present the same problem, solved in exactly the same way, in matrix notation;

$$\begin{bmatrix} 4 & -2 & 1 \\ -3 & -1 & 4 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \\ 13 \end{bmatrix}$$

- The arithmetic operations that we have performed *affect only the coefficients and the right-hand-side terms.*

- So we work with the matrix of coefficients *augmented* with the right-hand-side vector.

- We perform elementary row transformations to convert A to upper-triangular form:

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix},$$

$$\begin{array}{r} 3R_1 + 4R_2 \rightarrow \\ (-1)R_1 + 4R_3 \rightarrow \end{array}$$

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & -2 & 11 & 37 \end{bmatrix},$$

$$2R_2 - 10R_3 \rightarrow$$

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -10 & 19 & 77 \\ 0 & 0 & 72 & -216 \end{bmatrix} \Rightarrow$$

$$\begin{aligned} 4x_1 - 2x_2 + x_3 &= 15 \\ -10x_2 + 19x_3 &= 77 \\ -72x_3 &= -216 \end{aligned}$$

- The back-substitution step can be performed quite mechanically by solving the equations in reverse order. That is, $x_3 = 3, x_2 = -2, x_1 = 2$.

- Same solution with the non-matrix notation.
- Note that there exists the possibility that the set of equations has no solution, or that the prior procedure will fail to find it.
- During the triangularization step, if a zero is encountered on the diagonal, we cannot use that row to eliminate coefficients below that zero element.
- However, in that case, we can continue by **interchanging rows** and eventually achieve an upper-triangular matrix of coefficients.
- The real trouble is finding a zero on the diagonal after we have triangularized.
- If that occurs, the back-substitution fails, for we cannot divide by zero.
- It also means that the determinant is zero. There is no solution.

1.2.1 Gaussian Elimination

- The procedure just described has a major problem.
- While it may be satisfactory for hand computations with small systems, it is inadequate for a large system.
- Observe that the transformed coefficients can become very large (getting bigger!) as we convert to a triangular system.
- The method that is called *Gaussian elimination* avoids this by subtracting a_{i1}/a_{11} times the first equation from the i^{th} equation to make the transformed numbers in the first column equal to zero.
- We do similarly for the rest of the columns.
- Observe that zeros may be created in the diagonal positions even if they are not present in the original matrix of coefficients.
- A useful strategy to avoid (if possible) such zero divisors is to rearrange the equations so as to put the coefficient of largest magnitude on the diagonal at each step.
- This is called **pivoting**.

- Repeat the example of the previous section,

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ -3 & -1 & 4 & 8 \\ 1 & -1 & 3 & 13 \end{bmatrix},$$

$$\begin{array}{r} R_2 - (-3/4)R_1 \rightarrow \\ R_3 - (1/4)R_1 \rightarrow \end{array}$$

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -2.5 & 4.75 & 19.25 \\ 0 & -0.5 & 2.75 & 9.25 \end{bmatrix},$$

$$R_3 - (-0.5/-2.5)R_2 \rightarrow$$

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ 0 & -2.5 & 4.75 & 19.25 \\ 0 & 0 & 1.8 & 5.40 \end{bmatrix}$$

- The method we have just illustrated is called *Gaussian elimination*.
- In this example, **no pivoting was required** to make the largest coefficients be on the diagonal.
- Back-substitution, gives us $x_3 = 3,\ x_2 = -2,\ x_1 = 2$

Example m-file: Show steps in Gauss elimination and back substitution. No pivoting. (GEshow.m)

- We shall obtain answers that are just *close approximations to the exact answer* because of round-off error.
- When there are many equations, the effects of round-off (the term is applied to the error due to chopping as well as when rounding is used) may cause large effects.
- In certain cases, the coefficients are such that the results are particularly sensitive to round off; such systems are called **ill-conditioned**.

```
>> A=[4 -2 1 ; -3 -1 4 ; 1 -1 3];
>> b=[15 8 13];
>> GEshow(A,b')
Begin forward elimination with Augmented system:
     4    -2     1    15
    -3    -1     4     8
     1    -1     3    13
After elimination in column 1 with pivot = 4.000000
    4.0000   -2.0000    1.0000   15.0000
         0   -2.5000    4.7500   19.2500
         0   -0.5000    2.7500    9.2500
After elimination in column 2 with pivot = -2.500000
    4.0000   -2.0000    1.0000   15.0000
         0   -2.5000    4.7500   19.2500
         0         0    1.8000    5.4000

ans =
    2.0000
   -2.0000
    3.0000
```

- if we had stored the ratio of coefficients in place of zero (we show these in parentheses), our final form would have been

$$\begin{bmatrix} 4 & -2 & 1 & 15 \\ (-0.75) & -2.5 & 4.75 & 19.25 \\ (0.25) & (0.20) & 1.8 & 5.40 \end{bmatrix}$$

- The original matrix can be written as the product:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -0.75 & 1 & 0 \\ 0.25 & 0.20 & 1 \end{bmatrix}}_{L} * \underbrace{\begin{bmatrix} 4 & -2 & 1 \\ 0 & -2.5 & 4.75 \\ 0 & 0 & 1.8 \end{bmatrix}}_{U}$$

- This procedure is called a *LU-decomposition of A*.

- In this case,
$$A = L * U$$
where L is lower-triangular and U is upper-triangular.

- The determinant of two matrices, $B \times C$, is the product of each of the determinants, for this example we have
$$det(L * U) = det(L) * det(U) = det(U)$$
Because L is triangular and has only ones on its diagonal so that $det(L) = 1$.

- Thus, for our example, we have
$$det(A) = det(U) = (4) * (-2.5) * (1.8) = -18$$
since U is upper-triangular and its determinant is just the product of the diagonal elements.

- When there are row interchanges
$$det(A) = (-1)^m * u_{11} * \ldots * u_{nn}$$
where the exponent m represents *the number of row interchanges.*

- **Example**. Solve the following system of equations using Gaussian elimination.

$$\begin{array}{rrrr} & 2x_2 & & +x_4 = 0 \\ 2x_1 & +2x_2 & +3x_3 & +2x_4 = -2 \\ 4x_1 & -3x_2 & & x_4 = -7 \\ 6x_1 & +x_2 & -6x_3 & -5x_4 = 6 \end{array}$$

- In addition, obtain the determinant of the coefficient matrix and the LU decomposition of this matrix.

1. The augmented coefficient matrix is

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

2. We cannot permit a zero in the a_{11} position because that element is the pivot in reducing the first column.

3. We could interchange the first row with any of the other rows to avoid a zero divisor, but interchanging the first and fourth rows is our best choice. This gives

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

4 We again interchange before reducing the second column, not because we have a zero divisor, but because we want to preserve accuracy. Interchanging the second and third rows puts the element of largest magnitude on the diagonal.

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 1.6667 & 5 & 3.6667 & -4 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

5 Now we reduce in the second column

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 2.1818 & 3.3636 & -5.9999 \end{bmatrix}$$

6 No interchange is indicated in the third column. Reducing, we get

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ 0 & -3.6667 & 4 & 4.3333 & -11 \\ 0 & 0 & 6.8182 & 5.6364 & -9.0001 \\ 0 & 0 & 0 & 1.5600 & -3.1199 \end{bmatrix}$$

7 Back-substitution gives

$$x_1 = -0.50000, x_2 = 1.0000, x_3 = 0.33325, x_4 = -1.9999.$$

- The correct (exact) answers are $x_1 = -1/2, x_2 = 1, x_3 = 1/3, x_4 = -2$.
- In this calculation we have carried five significant figures and *rounded each calculation.*
- Even so, we do not have five-digit accuracy in the answers.
- The discrepancy is due to **round off**.
- **Example m-files:** (GEshow.m, GEPivShow.m)

```
>> A=[0 2 0 1; 2 2 3 2; 4 -3 0 1; 6 1 -6 -5];
>> b=[0 -2 -7 6];
>> GEshow(A,b')
Begin forward elimination with Augmented system:
     0      2      0      1      0
     2      2      3      2     -2
     4     -3      0      1     -7
     6      1     -6     -5      6
??? Error using ==> GEshow
zero pivot encountered
>> GEpivShow(A,b')
```

- In this example, if we had replaced the zeros below the main diagonal with the ratio of coefficients at each step, the resulting augmented matrix would be

$$\begin{bmatrix} 6 & 1 & -6 & -5 & 6 \\ (0.66667) & -3.6667 & 4 & 4.3333 & -11 \\ (0.33333) & (-0.45454) & 6.8182 & 5.6364 & -9.0001 \\ (0.0) & (-0.54545) & (0.32) & 1.5600 & -3.1199 \end{bmatrix}$$

- This gives a LU decomposition as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.66667 & 1 & 0 & 0 \\ 0.33333 & -0.45454 & 1 & 0 \\ 0.0 & -0.54545 & 0.32 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 1 & -6 & -5 \\ 0 & -3.6667 & 4 & 4.3333 \\ 0 & 0 & 6.8182 & 5.6364 \\ 0 & 0 & 0 & 1.5600 \end{bmatrix}$$

- It should be noted that the product of these matrices produces a permutation of the original matrix, call it A', where

$$A' = \begin{bmatrix} 6 & 1 & -6 & -5 \\ 4 & -3 & 0 & 1 \\ 2 & 2 & 3 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

- The determinant of the original matrix of coefficients can be easily computed according to the formula

$$det(A) = (-1)^2 * (6) * (-3.6667) * (6.8182) * (1.5600) = -234.0028$$

  which is close to the exact solution: -234.

- The exponent 2 is required, because there were two row interchanges in solving this system.

- To summarize

  1. The solution to the four equations
  2. The determinant of the coefficient matrix
  3. A LU decomposition of the matrix, A', which is just the original matrix, A, after we have interchanged its rows.

- "These" are readily obtained after solving the system by Gaussian elimination method.

Example m-file: LU factorization without pivoting. (luNopiv.m)
Example m-file: LU factorization with partial pivoting. (luPiv.m)

1.2.2 The Gauss-Jordan Method

- There are many variants to the Gaussian elimination scheme.

- One variant that is sometimes used is the *Gauss-Jordan* scheme.

  - In it, the elements above the diagonal are made zero at the *same time* that zeros are created below the diagonal.
  - Usually, the diagonal elements are made ones at the same time that the reduction is performed; this transforms the coefficient matrix into the identity matrix.

```
>> A=[0 2 0 1; 2 2 3 2; 4 -3 0 1; 6 1 -6 -5];
>> [L, U, P]=lu(A)
>> b = [ 0 -2 -7 6]
>> A\b
ans =
-0.5000
 1.0000
 0.3333
-2.0000
Remember
LUx=b
Ly=b --> find y first
Ux=y --> then find x
>> y=L\(P*b')
y =
    6.0000
  -11.0000
   -9.0000
   -3.1200
>> x=U\y
x =
   -0.5000
    1.0000
    0.3333
   -2.0000
```

- When this has been accomplished, the *column of right-hand sides* has been transformed into the solution vector.
- Pivoting is normally employed to preserve arithmetic accuracy.

$$(1)\begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 2 & -2 \\ 4 & -3 & 0 & 1 & -7 \\ 6 & 1 & -6 & -5 & 6 \end{bmatrix}$$

$$(2)\begin{bmatrix} 1 & 0.1667 & -1 & -0.8333 & 1 \\ 0 & 1.6667 & 5 & 3.3667 & -4 \\ 0 & -3.6667 & 4 & 4.3334 & -11 \\ 0 & 2 & 0 & 1 & 0 \end{bmatrix}$$

$$(3)\begin{bmatrix} 1 & 0 & -0.8182 & -0.6364 & 0.5 \\ 0 & 1 & -1.0909 & -1.1818 & 3 \\ 0 & 0 & 6.8182 & 5.6364 & -9 \\ 0 & 0 & 2.1818 & 3.3636 & -6 \end{bmatrix}$$

$$(4)\begin{bmatrix} 1 & 0 & 0 & 0.04 & -0.58 \\ 0 & 1 & 0 & -0.2308 & 1.56 \\ 0 & 0 & 1 & 0 & -1.32 \\ 0 & 0 & 0 & 1.5599 & -3.12 \end{bmatrix}$$

$$(5)\begin{bmatrix} 1 & 0 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 0 & 1.0001 \\ 0 & 0 & 1 & 0 & 0.3333 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

1. Interchanging rows 1 and 4, dividing the new first row by 6, and reducing the first column gives

2. Interchanging rows 2 and 3, dividing the new second row by -3.6667, and reducing the second column (operating above the diagonal as well as below) gives

3. No interchanges now are required. We divide the third row by 6.8182 and create zeros below and above.

4. We complete by dividing the fourth row by 1.5599 and create zeros above.

5. The fourth column is now the solution.

- While the Gauss-Jordan method might seem to require the same effort as Gaussian elimination, it really requires almost 50% more operations.