Chapter 23 - Gauss' Law

Simple way of solving complex problem. ⟸ due to symmetry
For certain charge distributions → use Gauss' law instead of considering the fields $d\vec{E}$.
$\underbrace{\text{Hypothetical closed surface}}$ enclosing charge distribution.

Gaussian Surface:
- can have any shape
- minimizes our calculations of $\vec{E}$
- mimics the symmetry of the charge distribution

Gauss' law relates the electric fields at points on a (closed) Gaussian surface to the net charge enclosed by that surface. SLN → Fig. 23-1 (OR reverse)

Flux

How much charge is enclosed? need a way → Flux (intercepted field)
Flux: product of an area and the field accross that area. SLN

Flux of an $\vec{E}$: Consider an arbitrary Gaussian surface immersed in a nonuniform electric field.

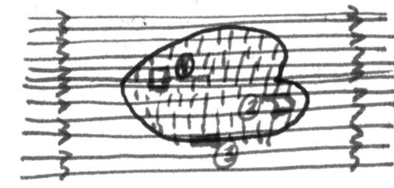
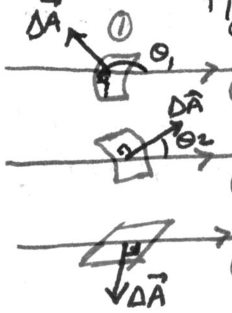


Fig. 23-3

Divide the surface into small flat squares each having area of ΔA. Since squares are very small electric field can be taken as constant.



$\vec{E}$ (inward) ~ negative
$\vec{E}$ (outward) ~ positive
$\vec{E}$ (parallel) ~ zero

$$\Phi = \sum \vec{E}\cdot\Delta\vec{A} \;\left\{\begin{matrix}\Delta A \text{ is} \\ \text{smaller} \\ \text{and} \\ \text{smaller}\end{matrix}\right\}\; \boxed{\Phi = \oint \vec{E}\cdot d\vec{A}}$$

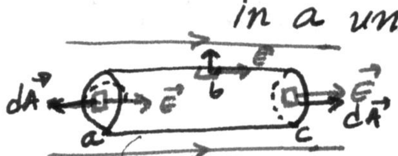
Electric flux through a Gaussian surface

closed surface

> The electric flux Φ through a Gaussian surface is proportional to the net number of electric field lines passing through that surface.

Example

Gaussian surface in the form of a cylinder of radius R immersed in a uniform $\vec{E}$. What is Φ?



Gaussian surface

$$\Phi = \oint \vec{E}\cdot d\vec{A} = \int_a \vec{E}\cdot d\vec{A} + \int_b \vec{E}\cdot d\vec{A} + \int_c \vec{E}\cdot d\vec{A}$$

$$= \int E(\cos 180)\,dA + \int E(\cos 90^\circ)\,dA + \int E(\cos 0)\,dA$$

$(A = \pi R^2)$

$$= -EA + 0 + EA$$

$\Phi = 0$: net flux zero $\left\{ \begin{matrix}\text{input} \\ \text{output}\end{matrix}\right.$ are same in magnitude, but opposite in direction.

SLN

Gauss' law

Relates the net flux Φ of on $\vec{E}$ through a closed surface (Gaussian surface) to the net charge q_{enc} → enclosed charge

$\varepsilon_0 \Phi = q_{enc}$ $\xrightarrow[\text{flux}]{\text{subs.}}$ $\boxed{\oint \vec{E}.d\vec{A} = \frac{q}{\varepsilon_0}}$

Gauss' law

- charge is placed in vacuum or air
- net charge < +, -, 0
- q_{enc} is (+) ⇒ Φ is outward
- q_{enc} is (-) ⇒ Φ is inword

SLN Fig. 23-6

S_1: $\vec{E}$ is outward → Φ is positive → net charge is (+) as Gauss' law

S_2: $\vec{E}$ is inward → Φ is negative → net charge is (-)

S_3: no charge is enclosed, $q_{enc}=0$ → $\Phi=0$: Field enter and also leave the surface

S_4: net charge is zero → $\Phi=0$ (net flux is zero)

What would happen if a large Q is placed neer the suface S_4?

→ The pattern of $\vec{E}$ lines would change, but flux would not change.

SLN Fig. 23-7 & Fig. 23-5

Gauss' Law and Coulomb's Law

Gauss' law, Coulombs law are diffrent ways of describing the relation btw $\vec{E}$ and q. SLN Fig. 23-8

⇒ So that, derive each from the other is possible.

Consider a positive charge $\xrightarrow[\text{surface}]{\text{a Gaussian}}$ $\varepsilon_0 \oint \vec{E}.d\vec{A} = q_{enc}$

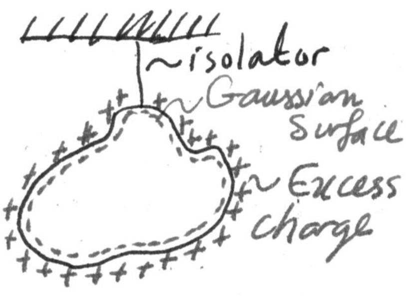
Gauss' law ⟺ Coulomb's law

a differential area from Gaussion surface (spherical).

$\left.\begin{array}{l} \vec{E} \parallel d\vec{A} \Rightarrow \theta = 0° \\ q_{enc} = q \end{array}\right\}$ $\varepsilon_0 \oint E dA = q \Rightarrow \left.\begin{array}{l} \varepsilon_0 E \oint dA = q \\ \varepsilon_0 E (4\pi r^2) = q \end{array}\right\}$ $\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$

A charged Isolated Conductor

Gauss' law permits us to provide } If an excess charge is placed on an isolated conductor, that amount of charge will move entirely to the suface of the conductor. None of these ~~exchange~~ excess charge will be found within the body of the conductor.



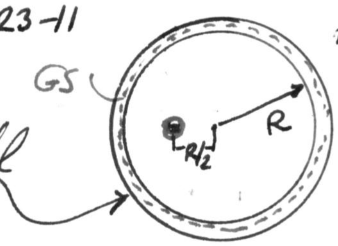
GS: Just below the surface $\vec{E} \to 0$ since net charge inside is 0

$\vec{E} \to 0$ again

Example Spherical metal shell, $\vec{E}$ and q_{enc}

A negatively charged particle is placed inside the neutral metal shell. Draw $\vec{E}$ lines and charge distribution.

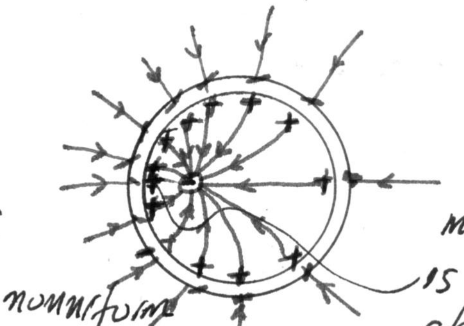
SLN Fig. 23-11



neutral metal shell

net charge inside GS is zero

$q = -5\mu C$



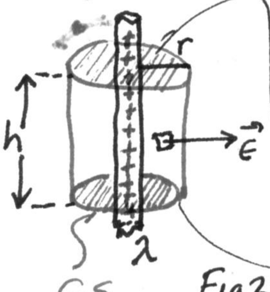
Total charge in shell is conserved

nonuniform

more positive charge is near negative charge.

Applying Gauss' law

1) Cylindrical Symmetry



GS Fig 23-12

- An infinetely long cylindrical plastic rod, ~~with a~~
- a uniform positive linear charge density, λ
- What is the magnitude of $\vec{E}$ at a distance r?
  - Directed radially outward. (+ charge)
  - Flux through end caps is zero since $\vec{E}$ is parallel to these caps. ($\vec{E} \perp d\vec{A}$)
  - Flux through cylindrical surface,

$$\Phi = EA\cos\theta = E(2\pi r h)\cos 0 = E(2\pi r h)$$

Gauss's law

$$\varepsilon_0 \Phi = q_{enc}$$

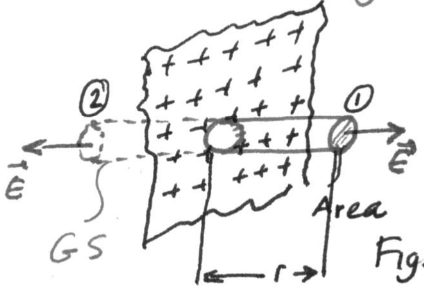
$$\varepsilon_0 E 2\pi r h = \lambda h$$

charge enclosed by GS

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$ line of charge

2) Planar Symmetry

2a Nonconducting Sheet

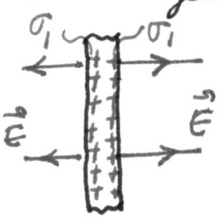


- Infinite nonconducting sheet
- a uniform positive surface charge density, σ
- What is the magnitude of $\vec{E}$ at a distance r?
  - Again cylindrical GS
  - Fig. 23-15 – Directed away from the sheet. (+ charge)
  - Flux through cylindrical surface is zero since $\vec{E} \parallel$ surface ($\vec{E} \perp d\vec{A}$)
  - Flux through end caps $\varepsilon_0(\underset{①}{EA} + \underset{②}{EA}) = \sigma A$

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$ sheet of charge

2b Two Conducting Plates Thin infinite conducting plate

(+) charged plate



(−) charged plate

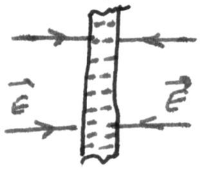
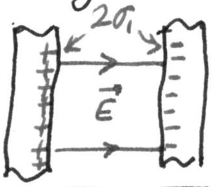


Fig. 23-16

They come close



- The excess charge on one plate will attract the excess charge on the other plate, and all the excess charge will move to the inner surfaces.
- charge density on each surface: $2\sigma_1$

$$\varepsilon_0(\underset{E_{q+}}{EA\cos 0} + \underset{E_{q-}}{EA\cos\theta}) = 2\sigma_1 A \Longrightarrow \boxed{E = \frac{2\sigma_1}{2\varepsilon_0} = \frac{\sigma}{\varepsilon_0}}$$

3) Spherical Symmetry

shell theorems SLN - Charged spherical shell, q
Radius, R

- Two Gaussian Surfaces GS1 & GS2
- GS2 $r \geq R$ (Remember shell theorem1)

Apply Gauss' law

$$\boxed{E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}}$$

Fig. 23-18

- GS1 $r < R$

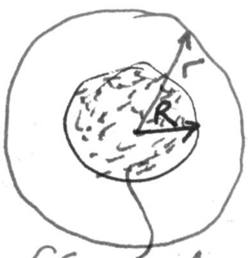
$\boxed{E=0}$

- Since GS1 encloses no charge
- Remember shell theorem2

$\vec{E}$ in Spherically Symmetric Charge Distribution

charge density, ρ. which is a function of only distance from center

- Entire charge lies within GS



enclosed charge is q. (ρ)

Fig. 23-19a

- $\vec{E}$: as if all the charge were a point charge at the center

- Entire charge doesn't lie within GS

Fig. 23-19b

q': the charge enclosed by GS

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q'}{r^2}$$

To find q': $\dfrac{q'}{\frac{4}{3}\pi r^3} = \dfrac{q}{\frac{4}{3}\pi R^3}$

$$\Rightarrow q' = q\frac{r^3}{R^3}$$

$$\Rightarrow \boxed{E = \frac{q}{4\pi\varepsilon_0 R^3}r}$$