Lecture 7
Some Discrete Probability Distributions I
Lecture Information

Ceng272 *Statistical Computations* at April 05, 2010

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Contents

Introduction and Motivation

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- **Binomial distribution**: test the effectiveness of a new drug.
- **Hypergeometric distribution**: test the number of defective items from a batch of production.
- **Negative binomial distribution (Geometric distribution)**: the number of trial on which the first success occurs. (next lecture)
- **Poisson distribution**: the number of outcomes occurring during a given time interval or in a specified region. (next lecture)

Discrete Uniform Distribution I

- **Discrete Uniform Distribution:** If the random variable X assumes the values $x_1, x_2, \ldots x_n$, with equal probabilities, then the discrete uniform distribution (probability mass function) is given by

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- **Example 5.1**: When a light bulb is selected at random from a box that contains a 40-watt bulb, a 60-watt bulb, a 75-watt bulb, and a 100-watt bulb, each element of the sample space $S = \{40, 60, 75, 100\}$ occurs with probability $1/4$.

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- **Example 5.2**: When a die is tossed, each element of the sample space $S = \{1, 2, 3, 4, 5, 6\}$ occurs with probability $1/6$.

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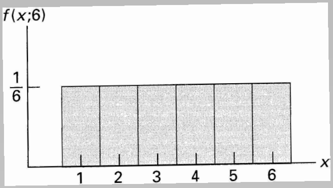

Figure: Histogram for the tossing of a die.

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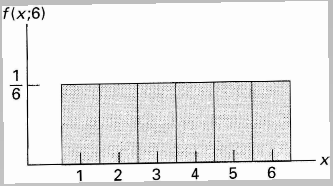

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- **Theorem 5.1:**:

The mean and variance of the discrete uniform distribution f(x; k) are

$$E(X) = \mu = \frac{1}{k}\sum_{i=1}^{k} x_i,\ \ \sigma_X^2 = \frac{1}{k}\sum_{i=1}^{k}(x_i - \mu)^2$$

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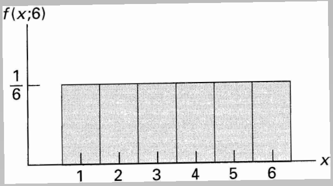

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- **Example 5.3**: Referring to Example 5.2 (tossing a die), we find that

$$\mu = 3.5,\quad \sigma^2 = 2.92$$

Binomial and Multinomial Distribution I

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$$X = \begin{cases} 1, & \textit{with probability } p, \\ 0, & \textit{with probability } 1 - p, \end{cases}$$

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- **The Bernoulli Probability Distribution**: The probability distribution of X is given by

$$P(X = x) = p(x) = p^x q^{1-x}, \; x = 0, 1$$

where p is called the parameter of Bernoulli probability distribution.

Binomial and Multinomial Distribution II

- Since X can be only 0 and 1;

  $E(X) = 1 * p + 0 * q = p, \sigma_X^2 = E(X^2) - [E(X)]^2 = p - p^2 = pq.$

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  - The repeated trials are **independent**.

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- The values of the binomial sums can be found in Table A.1

Binomial and Multinomial Distribution IV

TABLE A.1 (continued) Binomial Probability Sums $\sum_{x=0}^{r} b(x; n, p)$

n	r	p = 0.10	0.20	0.25	0.30	0.40	0.50	0.60	0.70	0.80	0.90
15	0	0.2059	0.0352	0.0134	0.0047	0.0005	0.0000				
	1	0.5490	0.1671	0.0802	0.0353	0.0052	0.0005	0.0000			
	2	0.8159	0.3980	0.2361	0.1268	0.0271	0.0037	0.0003	0.0000		
	3	0.9444	0.6482	0.4613	0.2969	0.0905	0.0176	0.0019	0.0001		
	4	0.9873	0.8358	0.6865	0.5155	0.2173	0.0592	0.0094	0.0007	0.0000	
	5	0.9978	0.9389	0.8516	0.7216	0.4032	0.1509	0.0338	0.0037	0.0001	
	6	0.9997	0.9819	0.9434	0.8689	0.6098	0.3036	0.0951	0.0152	0.0008	
	7	1.0000	0.9958	0.9827	0.9500	0.7869	0.5000	0.2131	0.0500	0.0042	0.0000
	8		0.9992	0.9958	0.9848	0.9050	0.6964	0.3902	0.1311	0.0181	0.0003
	9		0.9999	0.9992	0.9963	0.9662	0.8491	0.5968	0.2784	0.0611	0.0022
	10		1.0000	0.9999	0.9993	0.9907	0.9408	0.7827	0.4845	0.1642	0.0127
	11			1.0000	0.9999	0.9981	0.9824	0.9095	0.7031	0.3518	0.0556
	12				1.0000	0.9997	0.9963	0.9729	0.8732	0.6020	0.1841
	13					1.0000	0.9995	0.9948	0.9647	0.8329	0.4510
	14						1.0000	0.9995	0.9953	0.9648	0.7941
	15							1.0000	1.0000	1.0000	1.0000
16	0	0.1853	0.0281	0.0100	0.0033	0.0003	0.0000				
	1	0.5147	0.1407	0.0635	0.0261	0.0033	0.0003	0.0000			
	2	0.7892	0.3518	0.1971	0.0994	0.0183	0.0021	0.0001			
	3	0.9316	0.5981	0.4050	0.2459	0.0651	0.0106	0.0009	0.0000		
	4	0.9830	0.7982	0.6302	0.4499	0.1666	0.0384	0.0049	0.0003		
	5	0.9967	0.9183	0.8103	0.6598	0.3288	0.1051	0.0191	0.0016	0.0000	
	6	0.9995	0.9733	0.9204	0.8247	0.5272	0.2272	0.0583	0.0071	0.0002	

Figure: Binomial Probability Sums $B(r; n, p) = \sum_{x=0}^{r} b(x; n, p)$.

Binomial and Multinomial Distribution V

- **Example**: Suppose a professional basket player tries 5 free throws. The player is known to make 80% successful rate.

Binomial and Multinomial Distribution V

- **Example**: Suppose a professional basket player tries 5 free throws. The player is known to make 80% successful rate.
- Let X be the number of free throws he will make, then $X \sim b(n = 5, p = 0.8)$.

Table: Possible cases of 4 successes (o) and 1 miss (x) among 5 trials with $p = 0.8, q = 0.2$.

Trial	Possible Event	$p(x)$	Probability
1	oooox	ppppq	$(0.8)^4(0.2)^1 = 0.08192$
2	oooxo	pppqp	$(0.8)^4(0.2)^1 = 0.08192$
3	ooxoo	ppqpp	$(0.8)^4(0.2)^1 = 0.08192$
4	oxooo	pqppp	$(0.8)^4(0.2)^1 = 0.08192$
5	xoooo	qpppp	$(0.8)^4(0.2)^1 = 0.08192$

Since there are $\binom{5}{4} = 5$ cases of making 4 successes among 5 trials,

$$P(X = 4) = 5(0.8)^4(0.2)^1 = 0.4096.$$

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$$b(x;n,p) = b(2;4,\frac{3}{4}) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 =$$

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$$b(x;n,p) = b(2;4,\frac{3}{4}) = \binom{4}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^2 = \frac{4!}{2!2!} * \frac{3^2}{4^4} = \frac{27}{128}$$

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$$\begin{aligned}(q+p)^n &= \binom{n}{0}p^0q^n + \binom{n}{1}pq^{n-1} + \binom{n}{2}p^2q^{n-2} + \ldots + \binom{n}{n}p^nq^{n-n} \\ &= b(0;n,p) + b(1;n,p) + b(2;n,p) + \ldots + b(n;n,p) \\ &= 1\end{aligned}$$

since $p+q=1$

Binomial and Multinomial Distribution VII

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$$= 0.4032 - 0.2173 = 0.1859$$

Binomial and Multinomial Distribution VIII

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$$P(X \geq 1) = 1 - P(X = 0) = 1 - b(0; 20, 0.03) = 1 - 0.03^0 0.97^{20-0} = 0.4562$$

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Denote by Y the number of shipments containing at least one defective item; $b(y; 10, 0.4562)$

$$P(Y = 3) = \binom{10}{3} 0.4562^3 (1 - 0.4562)^{10-3} = 0.1602$$

Binomial and Multinomial Distribution IX

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  $$\mu = np \text{ and } \sigma^2 = npq$$

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- Solution: Example 5.5 was a binomial experiment with $n = 15$ and $p = 0.4$

$$\mu = np = 15 * 0.4 = 6$$

$$\sigma^2 = npq = 15 * 0.4 * 0.6 = 3.6 \Rightarrow \sigma = 1.897$$

The interval

$$\mu \pm 2\sigma = 6 \pm 2 * 1.897 \Rightarrow 2.206 \text{ to } 9.794$$

has a probability of at least $\frac{3}{4}$.

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$$P(X = 6) = \sum_{x=0}^{6} b(x; 10, 0.3) - \sum_{x=0}^{5} b(x; 10, 0.3)$$

$$= 0.9894 - 0.9527 = 0.0367$$

For values of $b(x; 10, 0.3)$, see Table A.1 from text book.

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- As a result, it is unlikely (3.6%chance) that 6 wells would be found impure if only 30% of all are impure. This casts considerable doubt on the conjecture and suggests that the impurity problem is much more severe.

Binomial and Multinomial Distribution XII

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Let X be a random variable with probability distribution $f(x)$. The mean of the random variable $g(X)$ is

$$f(x_1, x_2, \ldots, x_k; p_1, p_2, \ldots, p_k, n) = \binom{n}{x_1, x_2, \ldots, x_k} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

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- $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} p_i = 1$

  $$\binom{n}{x_1, x_2, \ldots, x_k} = \frac{n!}{x_1! x_2! \ldots x_k!}$$

  is the number of ways that yielding x_1 outcomes for E_1, x_2 outcomes for E_2, ..., x_k outcomes for E_k.

Binomial and Multinomial Distribution XIII

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- Solution:

Let X_i be the number of customers buy i^{th} product, $i = 1, 2, 3$. Then

$$f(5,4,1; \frac{60}{100}, \frac{30}{100}, \frac{10}{100}, 10) = \binom{10}{5,4,1} \frac{60}{100}^5 \frac{30}{100}^4 \frac{10}{100}^1 = 0.07936$$

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- Solution: Using the multinomial distribution , we have

$$f(2,1,3;\frac{2}{9},\frac{1}{6},\frac{11}{18},6) = \binom{6}{2,1,3}\frac{2}{9}^2\frac{1}{6}^1\frac{11}{18}^3$$

$$= \frac{6!}{2!1!3!} * \frac{2}{9}^2 * \frac{1}{6}^1 * \frac{11}{18}^3 = 0.1127$$

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$$\binom{k}{x}$$

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- So this plan is likely not desirable since it detects a bad lot (3 defectives) only about 30% of the time.

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The mean and variance of the hypergeometric distribution $h(x; N, n, k)$ are

$$\mu = \frac{nk}{N} \text{ and } \sigma^2 = \frac{N-n}{n-1} * n * \frac{k}{N} * \left(1 - \frac{k}{N}\right)$$

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- That is, at least three fourths of the time, the 5 components include less than 2 defectives.

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$$h(3; 5000, 10, 1000) \approx \sum_{x=0}^{3} b(x; 10, 0.2) - \sum_{x=0}^{2} b(x; 10, 0.2)$$

$$= 0.8791 - 0.6778 = 0.2013$$

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- **Multivariate HypergeometricDistribution**: If N items can be partitioned into the k cells $A_1, A_2, \dots, A_k$ with $a_1, a_2, \dots, a_k$ elements, respectively, then the probability distribution of the random variable $X_1, X_2, \dots, X_k$, representing the number of elements selected from $A_1, A_2, \dots, A_k$ in a random sample of size n, is

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- with $\sum_{i=1}^{k} x_i = n$ and $\sum_{i=1}^{k} a_i = N$

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- Solution:

$$f(1,2,2;3,4,3,10,5) = \frac{\binom{3}{1}\binom{4}{2}\binom{3}{2}}{\binom{10}{5}} = \frac{3}{14}$$